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Colorimetry and prime colours – a theorem

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Abstract. Human colour vision is the result of a complex process involving topics ranging from physics of light to perception. Whereas the diversity of light entering the eye in principle span an infinite-dimensional vector space in terms of the spectral power distributions, the space of human colour perceptions is three dimensional. One important consequence of this is that a variety of colours can be visually matched by a mixture of only three adequately chosen reference lights. It has been observed that there exists one particular set of monochromatic reference lights that, according to a certain definition, is optimal for producing colour matches. These reference lights are commonly denoted *prime colours*. In the present paper, we intend to rigorously show that the existence of prime colours is not particular to the human visual system as sometimes stated, but rather an *algebraic* consequence of the manner in which a kind of colorimetric functions called colour-matching functions are defined and transformed. The solution is based on maximisation of a determinant determining the gamut size of the colour space spanned by the prime colours. Cramer's rule for solving a set of linear equations is an essential part of the proof. By means of examples, it is shown that mathematically the optimal set of reference lights is not unique in general, and that the existence of a maximum determinant is not a necessary condition for the existence of prime colours.

1. Introduction

Human colour vision is the result of a complex process involving topics ranging from physics of light through the neurophysiology in the eye and brain, to perception. Colour sensation is thus more than a purely physical phenomenon. For instance, whereas the diversity of light entering the eye in principle span an infinite-dimensional vector space in terms of the spectral power distributions, the space of human colour perceptions is three dimensional (cf. Section 2).

One consequence of this is that a colour sensation can be described uniquely by three parameters. In communicating colours, we often use the terms "lightness", "hue" and "saturation". Another important consequence is that a variety of colours can be visually matched by a mixture of only three adequately chosen reference lights. In particular, when the reference lights are chosen as monochromatic light

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sources, a broad range of colours can be visually matched. Often, for certain colours evoked by one unit-power of monochromatic light, more than one unit-power of some of the reference lights is needed in order for a match to be achieved. It can be shown, however, that there exists one particular set of monochromatic reference lights for which no more than one unit-power of any of the reference lights is required for a visual match with any colour evoked by one unit-power of light [3,4]. The wavelengths of these particular monochromatic reference lights are commonly denoted prime colour wavelengths, or simply *prime colours*. A mathematical justification of the existence of prime colours is given by Brill [1,2] for a special case, using methods from calculus.

The purpose of the present paper is to rigorously show that the existence of prime colours is not particular to the human visual system, but rather an *algebraic* consequence of the manner in which a kind of colorimetric functions called colour-matching functions are defined and transformed. Section 2 gives a brief introduction to basic colorimetry sufficient for giving a mathematical presentation of the subject of prime colours, and in Section 3 a general theorem on prime colours is stated and proved. Some consequences of the theorem are briefly discussed in Section 4.

2. Colorimetry and prime colours

The origin of colorimetry is a type of psychophysical experiments known as *colour-matching experiments*:

Imagine two luminous fields lying side by side. Initially, one field, the target field, is of arbitrary colour. The colour of the other field results from superimposing (mixing) a number of lights of different relative spectral power distributions. A person with normal colour vision, the experimental *observer*, is given the task to adjust the intensities of these lights independently in order to make a colour match between the two fields in the sense that the two fields appear as one homogeneously coloured field. If a colour match cannot be achieved this way, the observer is allowed instead to mix one or more of the adjustable lights with the light of the target field, thereby trying to obtain a match. The initial colour of the target field is psychophysically determined by the intensity and positioning of each of the adjustable lights at colour match.

A *colour stimulus* is defined as radiant power of given magnitude and spectral composition, entering the eye and evoking the sensation of colour. Repeated colour-matching experiments show that for observers with normal colour vision the following holds true:

Property 1. Let $\mathbb{S}$ be a set of colour stimuli. If no pair of complementary subsets $\mathbb{T}$ and $\mathbb{U}$ of $\mathbb{S}$ exists for which some fractional mixture of the colour stimuli of $\mathbb{T}$ can be matched by some fractional mixture of the colour stimuli of $\mathbb{U}$, then $\mathbb{S}$ has at most three elements.

In visual science, this property of colour vision is called *trichromacy*. Also apparent from the colour-matching experiments is that the colour matches closely obey a set of rules, known as Grassman's laws of additive colour mixture:

*Property 2 (*Grassman's laws*).* Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be any four colour stimuli.

Symmetry: If $\mathcal{A}$ matches $\mathcal{B}$, then $\mathcal{B}$ matches $\mathcal{A}$ (i.e., symmetry with respect to which field each stimulus is originating from).

Transitivity: If $\mathcal{A}$ matches $\mathcal{B}$ and $\mathcal{B}$ matches $\mathcal{C}$, then $\mathcal{A}$ matches $\mathcal{C}$.

Proportionality: If $\mathcal{A}$ matches $\mathcal{B}$, then $k\mathcal{A}$ matches $k\mathcal{B}$, where k is any positive factor by which the radiant power of the colour stimulus is increased or reduced, while the relative spectral power distribution is kept the same.

Additivity: If any two of the three matches $\mathcal{A}$ matches $\mathcal{B}$, $\mathcal{C}$ matches $\mathcal{D}$ and $\mathcal{A}$ mixed with $\mathcal{C}$ matches $\mathcal{B}$ mixed with $\mathcal{D}$ hold good, then $\mathcal{A}$ mixed with $\mathcal{D}$ matches $\mathcal{B}$ mixed with $\mathcal{C}$ ("mixed with" here referring to superimposition of the radiant powers).

Properties 1 and 2 imply that the infinite variety of colour stimuli can be represented in a 3-dimensional vector space, the so-called *tristimulus space*. A vector in tristimulus space is termed *tristimulus vector*. In particular, any three colour stimuli $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ with the mutual property that none of them can be matched by a fractional mixture of the other two are represented by tristimulus vectors $\mathbf{R}$, $\mathbf{G}$ and $\mathbf{B}$ that are linearly independent and thus constitute a basis for the space. Hence, an arbitrary colour stimulus $\mathcal{Q}$ is represented by a tristimulus vector $\mathbf{Q}$ that is expressed by the vector equation

$$\mathbf{Q} = R_{\mathcal{Q}}\mathbf{R} + G_{\mathcal{Q}}\mathbf{G} + B_{\mathcal{Q}}\mathbf{B}. \tag{1}$$

Referring to colour matching, this equation should be read as "colour stimulus $\mathcal{Q}$ is matched by $R_{\mathcal{Q}}$ units of colour stimulus $\mathcal{R}$ mixed with $G_{\mathcal{Q}}$ units of colour stimulus $\mathcal{G}$ and $B_{\mathcal{Q}}$ units of colour stimulus $\mathcal{B}$". The coefficients $R_{\mathcal{Q}}$, $G_{\mathcal{Q}}$ and $B_{\mathcal{Q}}$ are called the *tristimulus values* of the colour stimulus $\mathcal{Q}$ relative to the *primary colour stimuli* or, in short form, the *primaries* $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$.[1] A negative tristimulus value indicates that, in order to achieve colour match, the proportion of the actual adjustable light is mixed with the initial light of the target field (corresponding to stimulus $\mathcal{Q}$).

In the particular case of a unit-power monochromatic colour stimulus e_λ of wavelength λ, the corresponding tristimulus vector $\mathbf{e}_\lambda$ is given by the vector equation

$$\mathbf{e}_\lambda = R_\lambda\mathbf{R} + G_\lambda\mathbf{G} + B_\lambda\mathbf{B}, \tag{2}$$

where the coefficients R_λ, G_λ and B_λ are termed *spectral tristimulus values*. By determining numerous colour matches, applying unit-power monochromatic colour stimuli e_{λ_i} $(i = 1, 2, \ldots, n)$ distributed over the entire visible spectrum in steps of just noticeable differences, functions can be derived that yield the spectral tristimulus values at given wavelengths. These functions – known as *colour-matching functions* – are by convention denoted $\bar{r}$, $\bar{g}$ and $\bar{b}$, referring to the primaries $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$, respectively. A particular set of colour-matching functions, which refer to unit-power monochromatic primaries of wavelengths $\lambda_{\mathcal{R}} = 645.2\,\text{nm}$, $\lambda_{\mathcal{G}} = 525.3\,\text{nm}$ and $\lambda_{\mathcal{B}} = 444.4\,\text{nm}$, is shown in Fig. 1.

Employing the colour-matching functions $\bar{r}$, $\bar{g}$ and $\bar{b}$, the tristimulus vectors of the different unit-power monochromatic colour stimuli are determined by a vector valued function $\mathbf{e}$ defined by the equation

[1] Primary colour stimuli (or primaries) should not be confused with prime colours.

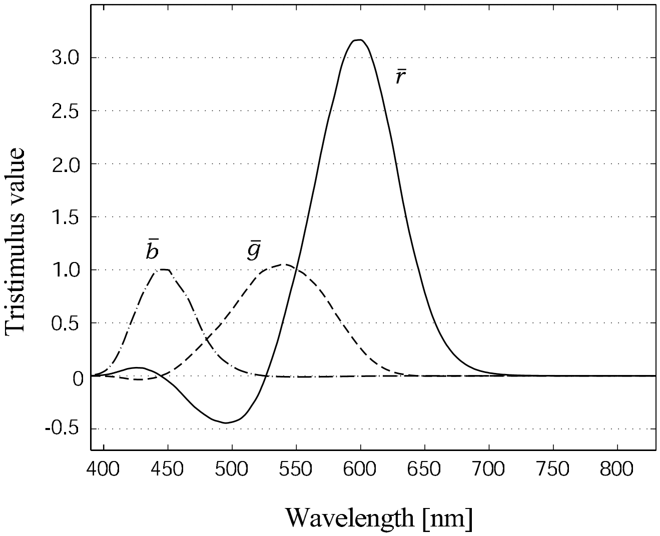

Fig. 1. The colour-matching functions of the Stiles–Burch 1959 observer. The functions $\bar{r}$, $\bar{g}$ and $\bar{b}$ refer to unit-power monochromatic primaries $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ of wavelengths $\lambda_{\mathcal{R}} = 645.2\,\text{nm}$, $\lambda_{\mathcal{G}} = 525.3\,\text{nm}$ and $\lambda_{\mathcal{B}} = 444.4\,\text{nm}$, respectively.

$$\mathbf{e}(\lambda) = \bar{r}(\lambda)\mathbf{R} + \bar{g}(\lambda)\mathbf{G} + \bar{b}(\lambda)\mathbf{B} \qquad (\lambda \in \omega), \tag{3}$$

where ω is the wavelength interval of the visible spectrum.

Moreover, letting $p_{\mathcal{Q}}$ denote the function describing the spectral power distribution of an arbitrary colour stimulus $\mathcal{Q}$, this implies that the monochromatic constituents of $\mathcal{Q}$ are represented by tristimulus vectors determined by a vector valued function $\mathbf{q}$ given as

$$\begin{aligned}\mathbf{q}(\lambda) &= p_{\mathcal{Q}}(\lambda)\mathbf{e}(\lambda) \\ &= p_{\mathcal{Q}}(\lambda)\bar{r}(\lambda)\mathbf{R} + p_{\mathcal{Q}}(\lambda)\bar{g}(\lambda)\mathbf{G} + p_{\mathcal{Q}}(\lambda)\bar{b}(\lambda)\mathbf{B} \qquad (\lambda \in \omega).\end{aligned} \tag{4}$$

As a consequence of Grassman's laws of additive colour mixture, the tristimulus vector of any colour stimulus can be calculated by summation over the tristimulus vectors of its monochromatic constituents. For the colour stimulus $\mathcal{Q}$, this is achieved by integrating $\mathbf{q}(\lambda)$ over the entire visible spectrum, ω. As the tristimulus vectors representing the primaries $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ are constant with respect to wavelength, the tristimulus vector of the colour stimulus $\mathcal{Q}$ is then determined to be

$$\begin{aligned}\mathbf{Q} = \int_{\omega} \mathbf{q}(\lambda)\,d\lambda = \int_{\omega} p_{\mathcal{Q}}(\lambda)\mathbf{e}(\lambda)\,d\lambda = \int_{\omega} p_{\mathcal{Q}}(\lambda)\bar{r}(\lambda)\,d\lambda\,\mathbf{R} \\ + \int_{\omega} p_{\mathcal{Q}}(\lambda)\bar{g}(\lambda)\,d\lambda\,\mathbf{G} + \int_{\omega} p_{\mathcal{Q}}(\lambda)\bar{b}(\lambda)\,d\lambda\,\mathbf{B},\end{aligned} \tag{5}$$

from which it follows that the tristimulus values of $\mathcal{Q}$ are given by the equations (cf. equation (1))

$$\begin{aligned} R_{\mathcal{Q}} &= \int_{\omega} p_{\mathcal{Q}}(\lambda)\bar{r}(\lambda)\,d\lambda, \\ B_{\mathcal{Q}} &= \int_{\omega} p_{\mathcal{Q}}(\lambda)\bar{g}(\lambda)\,d\lambda, \\ G_{\mathcal{Q}} &= \int_{\omega} p_{\mathcal{Q}}(\lambda)\bar{b}(\lambda)\,d\lambda. \end{aligned} \tag{6}$$

A further consequence of Grassman's laws is that the relationship between any two colour matches obtained from identical initial lights in the target field, but different sets of adjustable lights, may be derived by experimentally determining the basic colour matches between the two sets of adjustable lights themselves. If $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ are some chosen unit amounts of the stimuli originating from the lights of the one set, and $\mathcal{R}'$, $\mathcal{G}'$ and $\mathcal{B}'$ are some chosen unit amounts of the stimuli originating from the lights of the other, these basic colour matches are expressed in terms of the corresponding tristimulus vectors, $\mathbf{R}$, $\mathbf{G}$ and $\mathbf{B}$ and $\mathbf{R}'$, $\mathbf{G}'$ and $\mathbf{B}'$, by the vector equations

$$\begin{aligned} \mathbf{R}' &= R_{\mathcal{R}'}\mathbf{R} + G_{\mathcal{R}'}\mathbf{G} + B_{\mathcal{R}'}\mathbf{B}, \\ \mathbf{G}' &= R_{\mathcal{G}'}\mathbf{R} + G_{\mathcal{G}'}\mathbf{G} + B_{\mathcal{G}'}\mathbf{B}, \\ \mathbf{B}' &= R_{\mathcal{B}'}\mathbf{R} + G_{\mathcal{B}'}\mathbf{G} + B_{\mathcal{B}'}\mathbf{B}, \end{aligned} \tag{7}$$

where $R_{\mathcal{R}'}$, $G_{\mathcal{R}'}$ and $B_{\mathcal{R}'}$; $R_{\mathcal{G}'}$, $G_{\mathcal{G}'}$ and $B_{\mathcal{G}'}$; $R_{\mathcal{B}'}$, $G_{\mathcal{B}'}$ and $B_{\mathcal{B}'}$ are, respectively, the tristimulus values of the colour stimuli $\mathcal{R}'$, $\mathcal{G}'$ and $\mathcal{B}'$ relative to $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$.

Adopting $(\mathcal{R}, \mathcal{G}, \mathcal{B})$ and $\left(\mathcal{R}', \mathcal{G}', \mathcal{B}'\right)$ as two ordered sets of primaries (primary triples), it thus follows that, for an arbitrary colour stimulus $\mathcal{Q}$, the triple $\left(R'_{\mathcal{Q}}, G'_{\mathcal{Q}}, B'_{\mathcal{Q}}\right)$ of tristimulus values referring to the second primary triple can be calculated from the triple $\left(R_{\mathcal{Q}}, G_{\mathcal{Q}}, B_{\mathcal{Q}}\right)$ of tristimulus values referring to the first primary triple by treating the problem as an ordinary change of basis in tristimulus space. Accordingly, the transformation is given by the matrix equation

$$\begin{bmatrix} R'_{\mathcal{Q}} \\ B'_{\mathcal{Q}} \\ G'_{\mathcal{Q}} \end{bmatrix} = \begin{bmatrix} R_{\mathcal{R}'} & R_{\mathcal{G}'} & R_{\mathcal{B}'} \\ G_{\mathcal{R}'} & G_{\mathcal{G}'} & G_{\mathcal{B}'} \\ B_{\mathcal{R}'} & B_{\mathcal{G}'} & B_{\mathcal{B}'} \end{bmatrix}^{-1} \begin{bmatrix} R_{\mathcal{Q}} \\ B_{\mathcal{Q}} \\ G_{\mathcal{Q}} \end{bmatrix}. \tag{8}$$

Recalling that colour-matching functions yield the spectral tristimulus values of the unit-power monochromatic stimuli, this implies that the two ordered sets of colour-matching functions, $\left(\bar{r}, \bar{g}, \bar{b}\right)$ and $\left(\bar{r}', \bar{g}', \bar{b}'\right)$ – referring to the primary triples $(\mathcal{R}, \mathcal{G}, \mathcal{B})$ and $\left(\mathcal{R}', \mathcal{G}', \mathcal{B}'\right)$, respectively – are interrelated through the linear transformation

$$\begin{bmatrix} \bar{r}'(\lambda) \\ \bar{g}'(\lambda) \\ \bar{b}'(\lambda) \end{bmatrix} = \begin{bmatrix} R_{\mathcal{R}'} & R_{\mathcal{G}'} & R_{\mathcal{B}'} \\ G_{\mathcal{R}'} & G_{\mathcal{G}'} & G_{\mathcal{B}'} \\ B_{\mathcal{R}'} & B_{\mathcal{G}'} & B_{\mathcal{B}'} \end{bmatrix}^{-1} \begin{bmatrix} \bar{r}(\lambda) \\ \bar{g}(\lambda) \\ \bar{b}(\lambda) \end{bmatrix} \quad (\lambda \in \omega). \tag{9}$$

Moreover, if $\mathcal{R}'$, $\mathcal{G}'$ and $\mathcal{B}'$ are taken as unit-power monochromatic stimuli of wavelengths $\lambda_{\mathcal{R}'}$, $\lambda_{\mathcal{G}'}$ and $\lambda_{\mathcal{B}'}$, the columns of the matrix to be inverted will be constituted by the values of the colour-matching functions $\bar{r}$, $\bar{g}$ and $\bar{b}$ at these wavelengths. Hence, in this special case, the transformation equation reads:

$$\begin{bmatrix} \bar{r}'(\lambda) \\ \bar{g}'(\lambda) \\ \bar{b}'(\lambda) \end{bmatrix} = \begin{bmatrix} \bar{r}(\lambda_{\mathcal{R}'}) & \bar{r}(\lambda_{\mathcal{G}'}) & \bar{r}(\lambda_{\mathcal{B}'}) \\ \bar{g}(\lambda_{\mathcal{R}'}) & \bar{g}(\lambda_{\mathcal{G}'}) & \bar{g}(\lambda_{\mathcal{B}'}) \\ \bar{b}(\lambda_{\mathcal{R}'}) & \bar{b}(\lambda_{\mathcal{G}'}) & \bar{b}(\lambda_{\mathcal{B}'}) \end{bmatrix}^{-1} \begin{bmatrix} \bar{r}(\lambda) \\ \bar{g}(\lambda) \\ \bar{b}(\lambda) \end{bmatrix} \quad (\lambda \in \omega). \tag{10}$$

Clearly, for a primary colour stimulus, the tristimulus value referring to the primary itself is one, whereas the tristimulus values referring to the other two primaries are zero. For a colour-matching function referring to a monochromatic primary, this implies that its maximum value can never be less than unity. A typical example is the colour-matching functions of Fig. 1, which show to have values greater than one in certain wavelength intervals. With regard to the colour-matching experiment, this means that for certain unit-power monochromatic stimuli initially originating from the target field, more than one unit-power of the relevant primary is needed in order to obtain a colour match. As for the maximum power needed, the distinct maxima of the colour-matching functions of Fig. 1 indicate that this may differ between the three primaries.

Together these observations raise an interesting question: Is it possible to find a set of unit-power monochromatic primaries for which the maxima of the colour-matching functions are all equal to one? In terms of the mathematical nomenclature introduced above, this can be put as follows: Given a triple of colour-matching functions $(\bar{r}, \bar{g}, \bar{b})$, can particular values of $\lambda_{\mathcal{R}'}$, $\lambda_{\mathcal{G}'}$ and $\lambda_{\mathcal{B}'}$ be found that make equation (10) yield another triple of colour-matching functions $(\bar{r}', \bar{g}', \bar{b}')$ whose entries satisfy

$$\max_{\lambda \in \omega}(\bar{r}'(\lambda)) = \max_{\lambda \in \omega}(\bar{g}'(\lambda)) = \max_{\lambda \in \omega}(\bar{b}'(\lambda)) = 1\ ? \tag{11}$$

As shown in the next section, the answer to this question is that, indeed, on a certain criterion, such values of $\lambda_{\mathcal{R}'}$, $\lambda_{\mathcal{G}'}$ and $\lambda_{\mathcal{B}'}$ can be found. In addition to being of theoretical interest, this result also has significant practical consequences. For instance, it can be shown that a light source with monochromatic constituents of these wavelengths only, will be optimum when luminous efficiency and colour rendering are both taken into account [2,5].

3. A theorem on prime colours

Due to the colorimetry context, the main theorem of this article is formulated for $n = 3$ functions, but there is nothing in the proof that cannot easily be extended to an arbitrary integer $n \geq 1$. With reference to equation (10), the theorem can be stated as follows:

Theorem 1. *Let ω be an arbitrary set, and let $\bar{r}$, $\bar{g}$ and $\bar{b}$ be three arbitrary linearly independent real-valued functions defined on ω. For an arbitrary triple $\boldsymbol{\lambda} = (\lambda_{\mathcal{R}'}, \lambda_{\mathcal{G}'}, \lambda_{\mathcal{B}'}) \in \omega^3$ we define the matrix*

$$A_{\boldsymbol{\lambda}} = \begin{bmatrix} \bar{r}(\lambda_{\mathcal{R}'}) & \bar{r}(\lambda_{\mathcal{G}'}) & \bar{r}(\lambda_{\mathcal{B}'}) \\ \bar{g}(\lambda_{\mathcal{R}'}) & \bar{g}(\lambda_{\mathcal{G}'}) & \bar{g}(\lambda_{\mathcal{B}'}) \\ \bar{b}(\lambda_{\mathcal{R}'}) & \bar{b}(\lambda_{\mathcal{G}'}) & \bar{b}(\lambda_{\mathcal{B}'}) \end{bmatrix}. \tag{12}$$

If there exists a triple $\boldsymbol{\lambda}_p = (\lambda_{\mathcal{R}'_p}, \lambda_{\mathcal{G}'_p}, \lambda_{\mathcal{B}'_p}) \in \omega^3$ *such that*

$$\mathrm{Det}(A_{\boldsymbol{\lambda}_p}) = \max_{\boldsymbol{\lambda} \in \omega^3} \mathrm{Det}(A_{\boldsymbol{\lambda}}), \tag{13}$$

then the functions $\bar{r}'_p$*,* $\bar{g}'_p$ *and* $\bar{b}'_p$ *defined by the equation*

$$\begin{bmatrix} \bar{r}'_p(\lambda) \\ \bar{g}'_p(\lambda) \\ \bar{b}'_p(\lambda) \end{bmatrix} = \begin{bmatrix} \bar{r}(\lambda_{\mathcal{R}'_p}) & \bar{r}(\lambda_{\mathcal{G}'_p}) & \bar{r}(\lambda_{\mathcal{B}'_p}) \\ \bar{g}(\lambda_{\mathcal{R}'_p}) & \bar{g}(\lambda_{\mathcal{G}'_p}) & \bar{g}(\lambda_{\mathcal{B}'_p}) \\ \bar{b}(\lambda_{\mathcal{R}'_p}) & \bar{b}(\lambda_{\mathcal{G}'_p}) & \bar{b}(\lambda_{\mathcal{B}'_p}) \end{bmatrix}^{-1} \begin{bmatrix} \bar{r}(\lambda) \\ \bar{g}(\lambda) \\ \bar{b}(\lambda) \end{bmatrix} \tag{14}$$

satisfy

$$\max_{\lambda\in\omega}(\bar{r}'_p(\lambda)) = \max_{\lambda\in\omega}(\bar{g}'_p(\lambda)) = \max_{\lambda\in\omega}(\bar{b}'_p(\lambda)) = 1. \tag{15}$$

Proof. Since, by assumption, the functions $\bar{r}$, $\bar{g}$ and $\bar{b}$ are linearly independent there exists at least one triple $\boldsymbol{\lambda} = \left(\lambda_{\mathcal{R}'}, \lambda_{\mathcal{G}'}, \lambda_{\mathcal{B}'}\right) \in \omega^3$ that makes $A_{\boldsymbol{\lambda}}$ invertible. This can be shown by induction on the number n of functions. It is trivial for $n = 1$, because one linearly independent function is the same as one non-zero function. The induction step from $n = 2$ (which is easily generalised to any $n \geq 1$) is as follows: By the induction hypothesis there exists a pair (λ_2, λ_3) that makes the vectors $[\bar{g}(\lambda_2), \bar{b}(\lambda_2)]^T$ and $[\bar{g}(\lambda_3), \bar{b}(\lambda_3)]^T$ linearly independent. With this choice of (λ_2, λ_3) we use cofactor expansion along the first column to evaluate the determinant

$$\begin{aligned} L(\lambda) &= \begin{vmatrix} \bar{r}(\lambda) & \bar{r}(\lambda_2) & \bar{r}(\lambda_3) \\ \bar{g}(\lambda) & \bar{g}(\lambda_2) & \bar{g}(\lambda_3) \\ \bar{b}(\lambda) & \bar{b}(\lambda_2) & \bar{b}(\lambda_3) \end{vmatrix} \\ &= \begin{vmatrix} \bar{g}(\lambda_2) & \bar{g}(\lambda_3) \\ \bar{b}(\lambda_2) & \bar{b}(\lambda_3) \end{vmatrix} \bar{r}(\lambda) - \begin{vmatrix} \bar{r}(\lambda_2) & \bar{r}(\lambda_3) \\ \bar{b}(\lambda_2) & \bar{b}(\lambda_3) \end{vmatrix} \bar{g}(\lambda) + \begin{vmatrix} \bar{r}(\lambda_2) & \bar{r}(\lambda_3) \\ \bar{g}(\lambda_2) & \bar{g}(\lambda_3) \end{vmatrix} \bar{b}(\lambda). \end{aligned} \tag{16}$$

As apparent, $L(\lambda)$ is a linear combination of $\bar{r}(\lambda)$, $\bar{g}(\lambda)$ and $\bar{b}(\lambda)$, with the first coefficient non-zero by the induction hypothesis. By the assumption that the functions $\bar{r}$, $\bar{g}$ and $\bar{b}$ are linearly independent, this imply that $L(\lambda) \neq 0$ for at least one choice of λ. Choosing $\lambda = \lambda_1$ as a value for which $L(\lambda_1) \neq 0$ gives a non-singular, thus invertible, matrix.

By assumption, there exists a triple $\boldsymbol{\lambda}_p = \left(\lambda_{\mathcal{R}'_p}, \lambda_{\mathcal{G}'_p}, \lambda_{\mathcal{B}'_p}\right) \in \omega^3$ that maximises the value of the determinant, $\mathrm{Det}\left(A_{\boldsymbol{\lambda}}\right)$. Since a change in the ordering of the entries of the triple $\left(\lambda_{\mathcal{R}'}, \lambda_{\mathcal{G}'}, \lambda_{\mathcal{B}'}\right)$ results in a corresponding change in the ordering of the columns of the matrix $A_{\boldsymbol{\lambda}}$, and since an interchange of any two columns changes the sign, but not the absolute value, of $\mathrm{Det}\left(A_{\boldsymbol{\lambda}}\right)$, it follows that

$$\mathrm{Det}\left(A_{\boldsymbol{\lambda}_p}\right) = \max_{\boldsymbol{\lambda}\in\omega^3}\left(\mathrm{Det}\left(A_{\boldsymbol{\lambda}}\right)\right) = \max_{\boldsymbol{\lambda}\in\omega^3}\left|\mathrm{Det}\left(A_{\boldsymbol{\lambda}}\right)\right| > 0\,. \tag{17}$$

When both sides of equation (14) are multiplied from the left by $\boldsymbol{A}_{\boldsymbol{\lambda}_p}$, we obtain

$$\begin{bmatrix} \bar{r}(\lambda_{\mathcal{R}'_p}) & \bar{r}(\lambda_{\mathcal{G}'_p}) & \bar{r}(\lambda_{\mathcal{B}'_p}) \\ \bar{g}(\lambda_{\mathcal{R}'_p}) & \bar{g}(\lambda_{\mathcal{G}'_p}) & \bar{g}(\lambda_{\mathcal{B}'_p}) \\ \bar{b}(\lambda_{\mathcal{R}'_p}) & \bar{b}(\lambda_{\mathcal{G}'_p}) & \bar{b}(\lambda_{\mathcal{B}'_p}) \end{bmatrix} \begin{bmatrix} \bar{r}'_p(\lambda) \\ \bar{g}'_p(\lambda) \\ \bar{b}'_p(\lambda) \end{bmatrix} = \begin{bmatrix} \bar{r}(\lambda) \\ \bar{g}(\lambda) \\ \bar{b}(\lambda) \end{bmatrix}. \tag{18}$$

In interpreting this as a set of linear equations with $\bar{r}'_p(\lambda)$, $\bar{g}'_p(\lambda)$ and $\bar{b}'_p(\lambda)$ as the unknowns, Cramer's rule gives

$$\bar{r}'_p(\lambda) = \frac{\begin{vmatrix} \bar{r}(\lambda) & \bar{r}(\lambda_{\mathcal{G}'_p}) & \bar{r}(\lambda_{\mathcal{B}'_p}) \\ \bar{g}(\lambda) & \bar{g}(\lambda_{\mathcal{G}'_p}) & \bar{g}(\lambda_{\mathcal{B}'_p}) \\ \bar{b}(\lambda) & \bar{b}(\lambda_{\mathcal{G}'_p}) & \bar{b}(\lambda_{\mathcal{B}'_p}) \end{vmatrix}}{\begin{vmatrix} \bar{r}(\lambda_{\mathcal{R}'_p}) & \bar{r}(\lambda_{\mathcal{G}'_p}) & \bar{r}(\lambda_{\mathcal{B}'_p}) \\ \bar{g}(\lambda_{\mathcal{R}'_p}) & \bar{g}(\lambda_{\mathcal{G}'_p}) & \bar{g}(\lambda_{\mathcal{B}'_p}) \\ \bar{b}(\lambda_{\mathcal{R}'_p}) & \bar{b}(\lambda_{\mathcal{G}'_p}) & \bar{b}(\lambda_{\mathcal{B}'_p}) \end{vmatrix}} \tag{19}$$

and similar formulae for $\bar{g}'_p(\lambda)$ and $\bar{b}'_p(\lambda)$ – in the latter cases with the λ of the numerator occurring in the second and third column, respectively.

In the expressions for $\bar{r}'_p(\lambda)$, $\bar{g}'_p(\lambda)$ and $\bar{b}'_p(\lambda)$, also the numerators are determinants of matrices whose pattern equals that of $\boldsymbol{A}_{\boldsymbol{\lambda}}$. Hence, the absolute values of the numerators are less than or equal to the maximum value of the common denominator — this implying that the respective fractions have absolute value less than or equal to 1. Since, in particular, $\bar{r}'_p\left(\lambda_{\mathcal{R}'_p}\right) = \bar{g}'_p\left(\lambda_{\mathcal{G}'_p}\right) = \bar{b}'_p\left(\lambda_{\mathcal{B}'_p}\right) = 1$, the value 1 is, in fact, obtained, and we can thus conclude that the maximum values of the functions $\bar{r}'_p$, $\bar{g}'_p$ and $\bar{b}'_p$ are all equal to 1. □

Finding a triple $\boldsymbol{\lambda}_p$ that maximises Det $\left(\boldsymbol{A}_{\boldsymbol{\lambda}}\right)$ is not a trivial matter. If the functions $\bar{r}$, $\bar{g}$ and $\bar{b}$ are differentiable, one approach may be to set the partial derivatives equal to zero and then try to solve the resulting three non-linear equations in order to determine $\boldsymbol{\lambda}_p$. In most cases, however, this will show difficult. In practice, therefore, numerical methods of maximisation are used. In Fig. 2, the results of transforming the functions of Fig. 1 into colour-matching functions referring to monochromatic primaries of wavelengths equal to the corresponding prime colours are shown. For the resulting functions, $\bar{r}'_p$, $\bar{g}'_p$ and $\bar{b}'_p$, the arguments at peak values are calculated to be $\lambda_{\mathcal{R}'_p} = 600.0$ nm, $\lambda_{\mathcal{G}'_p} = 536.5$ nm and $\lambda_{\mathcal{B}'_p} = 447.9$ nm, respectively.

In the proof given above, there are no restrictions on the domain ω, but, as far as colorimetry is concerned, this will be the range of wavelengths of visible light. In practice, the functions are empirically given by function values on a finite subset of this wavelength domain. Since this imply that ω can be treated as a finite set, the problem of maximising the determinant is, in this case, reduced to a finite, thus solvable, task.

A determinant can be expressed as a polynomial of the matrix entries, and thus is a continuous function of those. Hence, if the functions $\bar{r}$, $\bar{g}$ and $\bar{b}$ are continuous, so is Det $(\boldsymbol{A}_{\boldsymbol{\lambda}})$ as a real-valued function on ω^3. If ω is taken as a closed interval of real numbers, this implies that maximum is obtained for at least one triple $\boldsymbol{\lambda}_p \in \omega^3$.

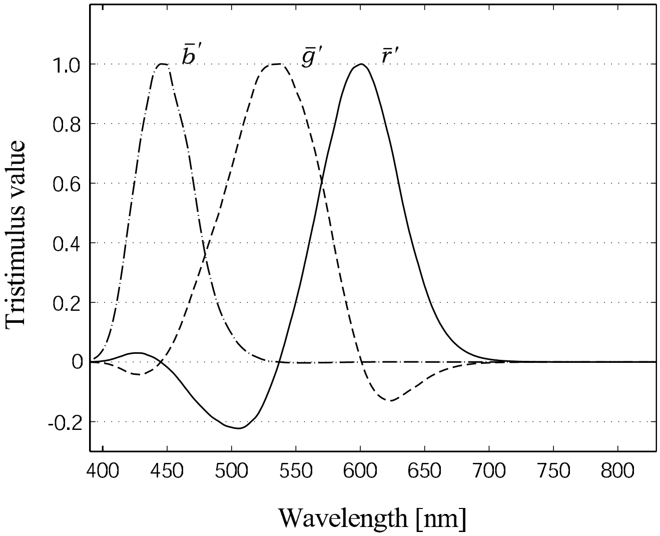

Fig. 2. The colour-matching functions $\bar{r}'_p$, $\bar{g}'_p$ and $\bar{b}'_p$ resulting from transferring the functions $\bar{r}$, $\bar{g}$ and $\bar{b}$ in Fig. 1. The peak values are at $\lambda_{\mathcal{R}'_p} = 600.0\,\text{nm}$, $\lambda_{\mathcal{G}'_p} = 536.5\,\text{nm}$ and $\lambda_{\mathcal{B}'_p} = 447.9\,\text{nm}$.

In particular, the maximum may need λ-values from the endpoints of ω. However, if the functions vanish to zero towards the endpoints, this is not the case, because then the determinant will also vanish to zero towards the border of ω^3. This is the situation discussed in [1]. A weaker restriction may be that the functions are "sufficiently small" towards the endpoints. What is meant by "sufficiently small" in this context can, if necessary, be determined in each individual case.

Experience from numerical investigations on colour-matching functions indicates that the maximum of Det $\left(\boldsymbol{A}_{\boldsymbol{\lambda}}\right)$ will exist. Furthermore, the triple $(\bar{r}'_p, \bar{g}'_p, \bar{b}'_p)$ is unique, except for a possible reordering. It is, however, possible to construct examples that embody more than one set of prime colours, or where prime colours exist, but the maximum of Det $\left(\boldsymbol{A}_{\boldsymbol{\lambda}}\right)$ does not. This will be shown in Propositions 2 and 3.

Proposition 1. *Let $\bar{\rho}$, $\bar{\gamma}$ and $\bar{\beta}$ be three functions that span the same linear space as $\bar{r}$, $\bar{g}$ and $\bar{b}$. Choose a triple $\boldsymbol{\lambda}_0 = \left(\lambda_{\mathcal{R}'_0}, \lambda_{\mathcal{G}'_0}, \lambda_{\mathcal{B}'_0}\right) \in \omega^3$ so that $\boldsymbol{A}_{\boldsymbol{\lambda}_0}$ is invertible. Then*

$$\begin{bmatrix} \bar{\rho}'_0(\lambda) \\ \bar{\gamma}'_0(\lambda) \\ \bar{\beta}'_0(\lambda) \end{bmatrix} = \begin{bmatrix} \bar{\rho}(\lambda_{\mathcal{R}'_0}) & \bar{\rho}(\lambda_{\mathcal{G}'_0}) & \bar{\rho}(\lambda_{\mathcal{B}'_0}) \\ \bar{\gamma}(\lambda_{\mathcal{R}'_0}) & \bar{\gamma}(\lambda_{\mathcal{G}'_0}) & \bar{\gamma}(\lambda_{\mathcal{B}'_0}) \\ \bar{\beta}(\lambda_{\mathcal{R}'_0}) & \bar{\beta}(\lambda_{\mathcal{G}'_0}) & \bar{\beta}(\lambda_{\mathcal{B}'_0}) \end{bmatrix}^{-1} \begin{bmatrix} \bar{\rho}(\lambda) \\ \bar{\gamma}(\lambda) \\ \bar{\beta}(\lambda) \end{bmatrix} = \begin{bmatrix} \bar{r}'_0(\lambda) \\ \bar{g}'_0(\lambda) \\ \bar{b}'_0(\lambda) \end{bmatrix}. \tag{20}$$

Proof. Let $\boldsymbol{B}_{\boldsymbol{\lambda}_0}$ be the matrix corresponding to $\boldsymbol{A}_{\boldsymbol{\lambda}_0}$, with $\bar{r}$, $\bar{g}$ and $\bar{b}$ replaced by $\bar{\rho}$, $\bar{\gamma}$ and $\bar{\beta}$, i.e., $\boldsymbol{B}_{\boldsymbol{\lambda}_0}$ is the matrix to be inverted in equation (20). Denote the column vector $[\bar{r}, \bar{g}, \bar{b}]^T$ by $\bar{\boldsymbol{r}}$ and the column vector $[\bar{\rho}, \bar{\gamma}, \bar{\beta}]^T$ by $\bar{\boldsymbol{\rho}}$. Let the transformed vectors $\boldsymbol{A}_{\boldsymbol{\lambda}_0}^{-1}\bar{\boldsymbol{r}}$ and $\boldsymbol{B}_{\boldsymbol{\lambda}_0}^{-1}\bar{\boldsymbol{\rho}}$ be denoted by $\bar{\boldsymbol{r}}'_0$ and $\bar{\boldsymbol{\rho}}'_0$, respectively, and let $\boldsymbol{K}$ be the transition matrix from the basis $\{\bar{\rho}, \bar{\gamma}, \bar{\beta}\}$ to the basis $\{\bar{r}, \bar{g}, \bar{b}\}$ so that $\bar{\boldsymbol{r}} = \boldsymbol{K}\bar{\boldsymbol{\rho}}$ and $\bar{\boldsymbol{\rho}} = \boldsymbol{K}^{-1}\bar{\boldsymbol{r}}$. Then

$$\bar{\rho}'_0 = \boldsymbol{B}_{\boldsymbol{\lambda}_0}^{-1}\bar{\boldsymbol{\rho}} = \boldsymbol{B}_{\boldsymbol{\lambda}_0}^{-1}\boldsymbol{K}^{-1}\bar{\boldsymbol{r}} = \left(\boldsymbol{K}\boldsymbol{B}_{\boldsymbol{\lambda}_0}\right)^{-1}\bar{\boldsymbol{r}}\,. \tag{21}$$

A fundamental property of any transformation given by equation (10) is that the values obtained by applying the transformed functions $\bar{r}'$, $\bar{g}'$ and $\bar{b}'$ on the entries of $\boldsymbol{\lambda}$ are given by the equation

$$\begin{bmatrix} \bar{r}'(\lambda_{\mathcal{R}'}) & \bar{r}'(\lambda_{\mathcal{G}'}) & \bar{r}'(\lambda_{\mathcal{B}'}) \\ \bar{g}'(\lambda_{\mathcal{R}'}) & \bar{g}'(\lambda_{\mathcal{G}'}) & \bar{g}'(\lambda_{\mathcal{B}'}) \\ \bar{b}'(\lambda_{\mathcal{R}'}) & \bar{b}'(\lambda_{\mathcal{G}'}) & \bar{b}'(\lambda_{\mathcal{B}'}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{22}$$

This can be seen by multiplying the column vectors $[1,0,0]^T$, $[0,1,0]^T$ and $[0,0,1]^T$ from the left by $\boldsymbol{A}_{\boldsymbol{\lambda}}$, this yielding the columns of $\boldsymbol{A}_{\boldsymbol{\lambda}}$. Multiplication of these columns from the left by $\boldsymbol{A}_{\boldsymbol{\lambda}}^{-1}$ subsequently leads to equation (22).

Since a linear transformation is uniquely determined from its behaviour on a basis, this also determines the transformation, and thus the corresponding transformation matrix, uniquely.

In particular,

$$\begin{bmatrix} \bar{\rho}'_0(\lambda_{\mathcal{R}'_0}) & \bar{\rho}'_0(\lambda_{\mathcal{G}'_0}) & \bar{\rho}'_0(\lambda_{\mathcal{B}'_0}) \\ \bar{\gamma}'_0(\lambda_{\mathcal{R}'_0}) & \bar{\gamma}'_0(\lambda_{\mathcal{G}'_0}) & \bar{\gamma}'_0(\lambda_{\mathcal{B}'_0}) \\ \bar{\beta}'_0(\lambda_{\mathcal{R}'_0}) & \bar{\beta}'_0(\lambda_{\mathcal{G}'_0}) & \bar{\beta}'_0(\lambda_{\mathcal{B}'_0}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

Consequently, by equation (21), equation (22) is true for the vector $\bar{\boldsymbol{r}}'$ given as $\left(\boldsymbol{K}\boldsymbol{B}_{\boldsymbol{\lambda}_0}\right)^{-1}\bar{\boldsymbol{r}}$, Moreover, equation (22) is true also if $\bar{\boldsymbol{r}}$ equals $\boldsymbol{A}_{\boldsymbol{\lambda}_0}^{-1}$. The uniqueness of the transformation thus leads to the conclusion that $\boldsymbol{A}_{\boldsymbol{\lambda}_0}^{-1} = \left(\boldsymbol{K}\boldsymbol{B}_{\boldsymbol{\lambda}_0}\right)^{-1}$ and, hence, that $\bar{\boldsymbol{\rho}}'_0 = \boldsymbol{A}_{\boldsymbol{\lambda}_0}^{-1}\bar{\boldsymbol{r}} = \bar{\boldsymbol{r}}'_0$.

In the proof, we have assumed that $\boldsymbol{B}_{\boldsymbol{\lambda}_0}$ is invertible if $\boldsymbol{A}_{\boldsymbol{\lambda}_0}$ is, which is, in fact, the case because $\boldsymbol{B}_{\boldsymbol{\lambda}_0} = \boldsymbol{K}^{-1}\boldsymbol{A}_{\boldsymbol{\lambda}_0}$ is a product of two matrices with non-zero determinants. □

Proposition 1 states that the transformed functions $\bar{r}'_0$, $\bar{g}'_0$ and $\bar{b}'_0$ are dependent on both the function space and the choice of $\boldsymbol{\lambda}_0 \in \omega^3$, but not on the choice of basis initially chosen for this space. In particular, the existence of triples $\boldsymbol{\lambda}_p \in \omega^3$ that give functions $\bar{r}'_p$, $\bar{g}'_p$ and $\bar{b}'_p$ with the property that $\max_{\lambda\in\omega}(\bar{r}'_p(\lambda)) = \max_{\lambda\in\omega}(\bar{g}'_p(\lambda)) = \max_{\lambda\in\omega}(\bar{b}'_p(\lambda)) = 1$, is a property of the space as such, independent of basis.

Experience from numerical investigations on colour-matching functions indicates that the triple $(\bar{r}', \bar{g}', \bar{b}')$ is unique, except for a possible reordering. In trying to establish this as a general theorem, the opposite conclusion emerged:

Proposition 2. *There exist functions $\bar{r}$, $\bar{g}$ and $\bar{b}$ that on transformation according to equation (10) yield more than one set $\{\bar{r}', \bar{g}', \bar{b}'\}$ with the property* $\max_{\lambda\in\omega}(\bar{r}'(\lambda)) = \max_{\lambda\in\omega}(\bar{g}'(\lambda)) = \max_{\lambda\in\omega}(\bar{b}'(\lambda)) = 1$. *Moreover, a triple of prime colours need not yield a maximum for* $\mathrm{Det}(\boldsymbol{A}_{\boldsymbol{\lambda}})$.

Proof. The proof will be made by an example:

Let $\omega = \{0, 1, 2, 3, 4\}$, and define the functions $\bar{r}$, $\bar{g}$ and $\bar{b}$ on ω according to the table

λ	0	1	2	3	4
$\bar{r}(\lambda)$	1	0	0	0	0
$\bar{g}(\lambda)$	0	$a\cos(\theta)$	$-\sin(\theta)$	1	0
$\bar{b}(\lambda)$	0	$a\sin(\theta)$	$\cos(\theta)$	0	1

(24)

with the parameters chosen so that $a > 1$, while $0 < |a\cos(\theta)| < 1$ and $0 < |a\sin(\theta)| < 1$ (e.g., $a = 1.1$ and $\theta = \pi/4$).

By choosing $\lambda_{\mathcal{R}'} = 0$, $\lambda_{\mathcal{G}'} = 3$ and $\lambda_{\mathcal{B}'} = 4$, the matrix $\boldsymbol{A}_{\boldsymbol{\lambda}}$ is the identity matrix, with $\mathrm{Det}\,(\boldsymbol{A}_{\boldsymbol{\lambda}}) = 1$. Then $\bar{r}' = \bar{r}$, $\bar{g}' = \bar{g}$ and $\bar{b}' = \bar{b}$, and according to the table, the maximum values of $\bar{r}'(\lambda)$, $\bar{g}'(\lambda)$ and $\bar{b}'(\lambda)$ are all equal to 1.

However, the choice $\lambda_{\mathcal{R}'} = 0$, $\lambda_{\mathcal{G}'} = 1$ and $\lambda_{\mathcal{B}'} = 2$ gives the maximum determinant:

$$\mathrm{Det}\left(\boldsymbol{A}_{\boldsymbol{\lambda}}\right) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & a\cos(\theta) & -\sin(\theta) \\ 0 & a\sin(\theta) & \cos(\theta) \end{vmatrix} = a > 1. \tag{25}$$

This second choice of $\lambda_{\mathcal{R}'}$, $\lambda_{\mathcal{G}'}$ and $\lambda_{\mathcal{B}'}$ gives a different set of functions $\{\bar{r}', \bar{g}', \bar{b}'\}$, but the maximum values of these functions are also all equal to 1. Furthermore, this shows that the first choice, $\lambda_{\mathcal{R}'} = 0$, $\lambda_{\mathcal{G}'} = 3$ and $\lambda_{\mathcal{B}'} = 4$, are prime colours that do not yield a maximum determinant. □

One might guess that the "if" in Theorem 1 could be replaced with an "if and only if", i.e., that the existence of a maximum determinant $\mathrm{Det}\,(\boldsymbol{A}_{\boldsymbol{\lambda}})$ is not only a sufficient, but also a necessary condition for the existence of functions $\bar{r}'$, $\bar{g}'$ and $\bar{b}'$ with maximum values 1. However, this is not the case in general. The example below showing this may seem quite artificial from a practical point of view, but is nevertheless included for the completeness of the mathematical treatment.

Proposition 3. *There exist functions $\bar{r}$, $\bar{g}$ and $\bar{b}$ such that the maximum value of* $\mathrm{Det}(A_\lambda)$ *is not obtained for any $\boldsymbol{\lambda} \in \omega^3$, but there is still a choice $\boldsymbol{\lambda} \in \omega^3$ such that* $\max(\bar{r}'(\lambda)) = \max(\bar{g}'(\lambda)) = \max(\bar{b}'(\lambda)) = 1$.

Proof. An example proving the proposition can be constructed by a modification of the functions in the proof of Proposition 2:

Extend the domain ω to include a small, open interval I containing 1, say $I = \{\lambda \,|\, 0.9 < \lambda < 1.1\}$. For $\lambda \in I$, let the function vary slightly, such that the maximum is not obtained in I, e.g., by defining $\bar{r}(\lambda) = 0$, $\bar{g}(\lambda) = \lambda\cos(\theta)$ and $\bar{b}(\lambda) = \lambda\sin(\theta)$ for $\lambda \in I$. The first choice in the proof of Proposition 2, $\lambda_{\mathcal{R}'} = 0$, $\lambda_{\mathcal{G}'} = 3$ and $\lambda_{\mathcal{B}'} = 4$, still gives functions $\bar{r}'$, $\bar{g}'$ and $\bar{b}'$ with maximum values equal to 1, provided $|\lambda\cos(\theta)| < 1$ and $|\lambda\sin(\theta)| < 1$ for all $\lambda \in I$ ($\theta = \pi/4$ can still be used). The alternative choice, $\lambda_{\mathcal{R}'} = 0$, $\lambda_{\mathcal{G}'} \in I$ and $\lambda_{\mathcal{B}'} = 2$, gives the determinant

$\mathrm{Det}\left(\boldsymbol{A}_{\boldsymbol{\lambda}}\right) = \lambda$. However, since the endpoint $\lambda = 1.1$ is not included in ω, the maximum is not obtained for any $\lambda \in \omega$. It is easy to check that neither is a maximum of $\mathrm{Det}\left(\boldsymbol{A}_{\boldsymbol{\lambda}}\right)$ obtained by any other choice of $\lambda_{\mathcal{R}'}$, $\lambda_{\mathcal{G}'}$ and $\lambda_{\mathcal{B}'}$. (For instance, if two or three λ-values are taken from the interval I, the determinants are zero.) The conclusion, therefore, is that even if a maximum of $\mathrm{Det}\left(\boldsymbol{A}_{\boldsymbol{\lambda}}\right)$ is not obtained for any triple $\boldsymbol{\lambda} \in \omega^3$, there may exist a transformation that yields functions $\bar{r}'$, $\bar{g}'$ and $\bar{b}'$ with maximum values all equal to 1. □

The examples of Proposition 2 and 3 can be adjusted to the set ω of wavelengths of visible light by transforming the argument values in these examples and letting the function values of the transformed arguments be as given in the table above while the function values of all other arguments are set equal to 0. By letting the functions decrease sufficiently fast to zero near the transformed values, it is easy also to construct extensions that are continuous except at the value transformed from 1.1.

4. Discussion

The proof of Theorem 1 given in Section 3 is, to our knowledge, not previously published. It is a constructive proof in the sense that the criterion of $\mathrm{Det}(\boldsymbol{A}_{\boldsymbol{\lambda}})$ being maximum emerges from the proof, a requirement unknown to us in advance. In searching the literature, we found a partial proof in [1]. This proof, which is a proof of existence only, is limited to continuously differentiable functions $\bar{r}$, $\bar{g}$ and $\bar{b}$ that vanish to zero at the border of ω. Neither does it follow from the proof that the triple $\boldsymbol{\lambda}_p$ that maximises $\mathrm{Det}\left(\boldsymbol{A}_{\boldsymbol{\lambda}}\right)$, is in fact the triple that solves our problem. The proof uses methods from calculus, which we from a philosophical point of view think obscures the primary reason for the truth of the theorem.

Within the field of colorimetry the quantity $\left|\mathrm{Det}\left(\boldsymbol{A}_{\boldsymbol{\lambda}}\right)\right|$ is known as the *gamut size*, a measure of the extent of the colour space spanned by a set of primaries, in this case a set of monochromatic primaries of wavelengths $\boldsymbol{\lambda}$. The fact that maximum gamut size corresponds to functions satisfying $\max_{\lambda\in\omega}(\bar{r}'(\lambda)) = \max_{\lambda\in\omega}(\bar{g}'(\lambda)) = \max_{\lambda\in\omega}(\bar{b}'(\lambda)) = 1$, is empirically justified. Theoretically, this property is equivalent to Theorem 1. The a priori assumption that maximum gamut size is associated with functions with this property, leads to an alternative way of proving Theorem 1. In Reference [2], Section 4 (p. 38), this is done for the case of the domain ω being finite. The proof given there can be reformulated into an alternate proof of the more general theorem. The proof is a contrapositive proof, that we, in short, formulate as follows:

Suppose that $\bar{r}'_p$, $\bar{g}'_p$ and $\bar{b}'_p$ are obtained by applying the transformation matrix $\boldsymbol{A}_{\boldsymbol{\lambda}_p}^{-1}$, where $\boldsymbol{\lambda}_p = (\lambda_{\mathcal{R}'p}, \lambda_{\mathcal{G}'p}, \lambda_{\mathcal{B}'p})$ is chosen so that $\mathrm{Det}(\boldsymbol{A}_{\boldsymbol{\lambda}_p})$ is maximum. Suppose also that at least one of the functions has a value greater than 1, say $\bar{b}'_p(\lambda_1) > 1$. Then we can transform once more by the same method, replacing $\bar{r}$, $\bar{g}$ and $\bar{b}$ by $\bar{r}'_p$, $\bar{g}'_p$ and $\bar{b}'_p$ as the initial functions and using $\boldsymbol{\lambda}_1 = (\lambda_{\mathcal{R}'p}, \lambda_{\mathcal{G}'p}, \lambda_1)$. Denoting by $\boldsymbol{B}_{\boldsymbol{\lambda}_1}$ the matrix corresponding to $\boldsymbol{A}_{\boldsymbol{\lambda}}$, $\mathrm{Det}(\boldsymbol{B}_{\boldsymbol{\lambda}_1}) = \bar{b}'_p(\lambda_1) > 1$. The composite transformation obtained by multiplying $[\bar{r}, \bar{g}, \bar{b}]^T$ from the left by the matrix

$$B_{\boldsymbol{\lambda}_1}^{-1} A_{\boldsymbol{\lambda}_p}^{-1} = \left(A_{\boldsymbol{\lambda}_p} B_{\boldsymbol{\lambda}_1}\right)^{-1} \tag{26}$$

equals the transformation obtained by initially choosing $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1$, i.e., $A_{\boldsymbol{\lambda}_1} = A_{\boldsymbol{\lambda}_p} B_{\boldsymbol{\lambda}_1}$. (This follows by an argument similar to that given at the end of the proof of Proposition 1.) Consequently,

$$\mathrm{Det}(A_{\boldsymbol{\lambda}_1}) = \mathrm{Det}(A_{\boldsymbol{\lambda}_p})\mathrm{Det}(B_{\boldsymbol{\lambda}_1}) = \mathrm{Det}(A_{\boldsymbol{\lambda}_p})\, \bar{b}'_p(\lambda_1) > \mathrm{Det}(A_{\boldsymbol{\lambda}_p})\,. \tag{27}$$

This, however, contradicts the assumption that $\mathrm{Det}(A_{\boldsymbol{\lambda}_\mathrm{p}})$ is maximum.

By Proposition 2, a triple $\boldsymbol{\lambda}_p$ yielding functions with maximum value 1 is not in general unique. For a real-world tristimulus space, however, empirical evidence suggests that they are. The triple $\boldsymbol{\lambda}_p = (\lambda_{\mathcal{R}'_p}, \lambda_{\mathcal{G}'_p}, \lambda_{\mathcal{B}'_p})$ maximising Det $(A_{\boldsymbol{\lambda}})$ is then, by Proposition 1, a property of the tristimulus space itself. The triple $\boldsymbol{\lambda}_p$ is not dependent on the basis, and, hence, not on the actual sources initially used in the psychophysical determination of the functions $\bar{r}$, $\bar{g}$ and $\bar{b}$. In this case, $\boldsymbol{\lambda}_p$ is determined numerically from these functions. The results vary due to experimental uncertainty, and Thornton [3,4] reports the approximate values $\lambda_{\mathcal{R}_p} \approx 610\,\mathrm{nm}$, $\lambda_{\mathcal{G}_p} \approx 530\,\mathrm{nm}$ and $\lambda_{\mathcal{B}_p} \approx 450\,\mathrm{nm}$. (The reason that these values are somewhat different from the ones reported in Section 3, is that the colour-matching functions of the Stiles–Burch 1959 observer are different from those used by Thornton.) These values are well known as the primary colour wavelengths, empirically discovered and put into practical use long before this mathematical justification.

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