*General Relativity and Gravitation, Vol. 28, No. 4, 1996*

# Vacuum Energy and Inertial Dragging

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*Received March 28, 1995. Rev. version September 26, 1995*

We investigate if there is any inertial dragging effect associated with vacuum energy. Spacetime inside and outside a rotating thin shell, as well as the mechanical properties of the shell, are analyzed by means of Israel's general relativistic theory of surface layers. Our investigation generalizes that of Brill and Cohen, who found vacuum-solutions of Einstein's field equations (with vanishing cosmological constant), inside and outside a rotating shell. We include a nonvanishing vacuum-energy inside the shell. It is found that the inertial dragging angular velocity increases with increasing density of vacuum energy.

## 1. INTRODUCTION

Vacuum-energy has some remarkable, and well-known, gravitational properties. Together with the Lorentz invariance of vacuum Einstein's field equations imply that vacuum acts upon itself with repulsive gravity. It seems, however, that the inertial dragging properties of vacuum have not been investigated. The reason may be that inertial dragging, i.e. the Lense–Thirring effect, is usually associated with rotational motion, and the Lorentz invariance of the vacuum implies that vacuum-energy is non-rotating. However, it *has* sense to say that vacuum-energy has expansion. At the end of the inflationary era this expansion is transferred to radiation and matter, and this may explain the observed state of expansion of the universe. Likewise the observed absence of cosmic rotation may be

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due to the non-rotation of the vacuum-energy in the vacuum-dominated inflationary era [1].

In this article we investigate the possibility that the vacuum-energy may have determined the large scale motion of the universe in a Machian way, by means of a mechanism involving the inertial dragging effect. This possibility depends upon the answer to the question: "Is there any inertial dragging effect associated with vacuum energy?" In order to approach this question we generalize the classical investigations of Lense and Thirring [2] and later of Brill and Cohen [3,4], where they established the existence of the rotational inertial dragging effect, and noted its possible cosmic significance.

## 2. INERTIAL DRAGGING INSIDE A ROTATING SPHERICAL SHELL

Brill and Cohen [3] found the spacetime line-element inside and outside a thin shell rotating with angular velocity $\omega_S$. They expressed the line-element in isotropic coordinates and found

$$ds^2 = V^2 dt^2 - \psi^4 [d\bar{r}^2 + \bar{r}^2 \{d\theta^2 + \sin^2\theta (d\phi - \Omega(\bar{r}) dt)^2\}] \tag{1}$$

with

$$V = \begin{cases} (\bar{r} - M/2)/(\bar{r} + M/2), & \bar{r} > \bar{r}_0 \\ (\bar{r}_0 - M/2)/(\bar{r}_0 + M/2), & \bar{r} < \bar{r}_0 \end{cases} \tag{2}$$

$$\psi = \begin{cases} 1 + M/2r, & \bar{r} > \bar{r}_0 \\ 1 + M/2\bar{r}_0, & \bar{r} < \bar{r}_0 \end{cases} \tag{3}$$

where $M$ and $\bar{r}_0$ are the mass and radius of the shell, respectively. The function $\Omega(\bar{r})$ is the angular velocity of the inertial dragging field. It is given by

$$\Omega(\bar{r}) = \begin{cases} \Omega_B (\bar{r}_0 \psi_0^2 / \bar{r} \psi^2)^3, & \bar{r} > \bar{r}_0 \\ \Omega_B\,, & \bar{r} < \bar{r}_0 \end{cases} \tag{4}$$

where

$$\Omega_B = \omega_S \Big/ \left(1 + \frac{3(\bar{r}_0 - M/2)^2}{4M(\bar{r}_0 - M/4)}\right). \tag{5}$$

This result has some interesting properties. First of all it reduces to the result

$$\Omega_B = \frac{4M\omega_S}{3\bar{r}_0} \tag{6}$$

of Lense and Thirring [2] in the weak field limit, i.e. to first order in $M/\bar{r}_0$. Also it was noted by Brill and Cohen [4] that 'perfect dragging', i.e.

$\Omega_B = \omega_S$ — the induced rotation inside the shell equals the rotation of the shell — is possible. It occurs if the shell is positioned at its Schwarzschild radius, $\bar{r}_0 = M/2$. A shell of matter of radius equal to its Schwarzschild radius may be taken as an idealized model of our universe. In such a model there cannot be a rotation of the local inertial frames relative to the large masses in the universe. The result of Brill and Cohen may thus explain, in a Machian way, the observation that the swinging plane of a Foucault pendulum docs not rotate relative to the stars.

In the present article we generalize the investigation of Brill and Cohen by including vacuum energy inside the rotating shell.

## 3. VACUUM ENERGY

There is now a great literature on inflationary cosmological models [5,6]. In most of these models the vacuum energy is due to a scalar field $\phi$ with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi) \tag{7}$$

where the potential $V(\phi)$ is typically of the Coleman–Weinberg form for the $SU_5$ Higgs field. The energy-momentum tensor for this field is

$$T_{\mu\nu} = -\phi_{,\mu}\phi_{,\nu} - \mathcal{L}g_{\mu\nu}\,. \tag{8}$$

Assuming that the universe model is spatially homogeneous, $T_{\mu\nu}$ takes the perfect fluid form with energy density and pressure given by

$$\rho = \langle\tfrac{1}{2}\dot{\phi}^2\rangle + V(\phi), \qquad p = \langle\tfrac{1}{2}\dot{\phi}^2\rangle - V(\phi). \tag{9}$$

A perfect fluid with Lorentz invariant properties shall here be called a 'vacuum fluid'. Demanding that the components of the energy-momentum tensor be Lorentz invariant leads to the form [7]

$$T_{\mu\nu} = \rho_V g_{\mu\nu} \tag{10}$$

where the energy density $\rho_V$ is in general a function of the four spacetime coordinates. In a spatially homogeneous universe model the density measured by an observer can at most depend upon the time coordinate. Due to the relativity of simultaneity the homogeneity of space is Lorentz invariant only if $\rho_V$ = constant.[2] In the present work we shall consider a universe model which fulfills this condition.

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[2] Editor's note: Equation (10) and the conservation law $T^{\mu\nu}{}_{;\nu} = 0$ already imply $\rho_V$ = constant without further conditions.

Comparing the form (10) of the energy-momentum tensor with that of a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \tag{11}$$

we find that the equation of state for a vacuum fluid is

$$p_V = -\rho_V \, . \tag{12}$$

As to the effect of the scalar field upon the spacetime geometry, this field acts as a combination of a vacuum fluid and a Zel'dovich fluid (a stiff fluid with sound velocity equal to the velocity of light) with equation of state

$$p_Z = \rho_Z = \langle \tfrac{1}{2}\dot{\phi} \rangle. \tag{13}$$

General relativistic models of spacetimes with vacuum energy inside and outside nonrotating spherical singular shells have been thoroughly investigated [8–12]. The effect of the vacuum energy outside a static shell is only to modify the equation of state for the matter that the shell consists of. In our approach, based upon a perturbation of the static situation, and for our intention, which is to investigate the effect of the vacuum energy inside a rotating shell (i.e. inside our universe) upon the motion of inertial frames in this region, it will be suitable to assume vanishing vacuum energy outside the shell. Also the Zel'dovich component of the energy-momentum vanishes in a stationary situation, which is considered in the present investigation.

## 4. ROTATING SHELL CONTAINING VACUUM ENERGY

In the static case there is de Sitter spacetime inside the shell and Schwarzschild spacetime outside it. The shell is assumed to rotate slowly, and the effect of the rotation is introduced as a perturbation of the static metric. The line-element is now written

$$ds^2 = g(r)dt^2 - \frac{dr^2}{f(r)} - r^2\{d\theta^2 + \sin^2\theta(d\phi - \Omega(r)dt)^2\} \tag{14}$$

where

$$f(r) = \begin{cases} 1 - 2M/r \equiv \beta_S^2 \, , & r > r_0 \\ 1 - 8\pi\rho r^2/3 \equiv \beta_D^2 \, , & r < r_0 \end{cases} \tag{15}$$

and

$$g(r) = \begin{cases} \beta_S^2 \, , & r > r_0 \\ (\beta_{S_0}\beta_D/\beta_{D_0})^2 , & r < r_0 . \end{cases} \tag{16}$$

Here $\rho$ is the constant energy density of vacuum, and the subscript 0 indicates the value at the shell, i.e. at $r = r_0$. The mass $M$ inside $r = r_0$ is partly due to the vacuum energy, and partly to the shell.

Both outside and inside the shell the 03-component of Einstein's field equations reduces to

$$r\Omega'' + 4\Omega' = 0\,. \tag{17}$$

Integrating this equation, demanding non-singularity at $r = 0$, vanishing dragging angular velocity at infinity, and continuity across the shell, we obtain

$$\Omega = \begin{cases} \Omega_P (r_0/r)^3, & r > r_0 \\ \Omega_P & r < r_0\,. \end{cases} \tag{18}$$

The constant $\Omega_P$ will now be calculated by means of Israel's general relativistic theory [13] of singular surfaces in terms of the radius, mass and angular velocity of the shell.

Let $V^-$ and $V^+$ be the spaces inside and outside the shell, $\Sigma$, respectively. The way $\Sigma$ curves in $V^-$ or $V^+$ is described by the *extrinsic curvature tensor* with components

$$K_{ij} = -n_{i;j} = -n_{i,j} + n_k \Gamma^k{}_{ij} \tag{19}$$

where $n$ is the unit normal vector of $\Sigma$. Let $K^+_{ij}$ denote the value of $K_{ij}$ in $V^+$, and $K^-_{ij}$ its value in $V^-$. Introduce $K^\pm = g^{ij} K^\pm_{ij}$ and

$$[K_{ij}] = K^+_{ij} - K^-_{ij}, \qquad [K] = K^+ - K^-\,. \tag{20}$$

The energy-momentum tensor of the shell is given by

$$S_{ij} = -\frac{1}{8\pi}\left( [K_{ij}] - g_{ij}[K] \right). \tag{21}$$

The components of the energy-momentum tensor may be interpreted physically by the following procedure. The eigenvalues $\lambda_{(k)}$ and eigenvectors $v_{(k)}$ of this tensor are given by

$$|S^i{}_j - \lambda_{(k)} \delta^i{}_j| = 0 \tag{22}$$

and

$$S^i{}_j v^j_{(k)} = \lambda_{(k)} v^j_{(k)} \tag{23}$$

where the vertical bars denote the determinant, and the subscript in the parenthesis is only a tag telling 'which vector' and does not denote the component of a vector. Equation (22) represents a third degree equation for $\lambda$ with roots $\lambda_{(t)}$, $\lambda_{(\theta)}$, $\lambda_{(\phi)}$. Equation (23) gives the three associated eigenvectors $v_{(t)}$, $v_{(\theta)}$, $v_{(\phi)}$. It follows from eq. (23) and the symmetry of the energy-momentum tensor that they are orthogonal. They are fixed by choosing them to be unit vectors. These vectors then represent an orthonormal basis-field at the source. The components of $S$ are now given the following physical interpretation. The vector $v_{(t)} = u$ is the four-velocity field of the shell. The eigenvalue $\lambda_{(t)} = \sigma$ is the energy-density as measured by an observer comoving with the shell. The eigenvalues $\lambda_{(k)} = -p_{(k)}$, $k = 1, 2$ are the negative of the stresses he measures.

We shall later need the following decomposition of the energy-momentum tensor of the shell

$$S^i{}_j = \sigma u^i u_j + \sum_{k=1}^{2} p_{(k)} v^i_{(k)} v_{(k)_j} \, . \tag{24}$$

Calculating the components of the energy-momentum tensor of the shell from eq. (14) and the line-elements in eqs. (14)–(16), we find

$$S^t{}_t = \frac{\beta_{D_0} - \beta_{S_0}}{4\pi r_0} \, , \tag{25}$$

$$S^\theta{}_\theta = S^\phi{}_\phi = \left\{ \beta_{D_0} - \beta_{S_0} - \left( \frac{M}{r_0 \beta_{S_0}} + \frac{8\pi \rho r_0^2}{3\beta_{D_0}} \right) \right\} , \tag{26}$$

$$S^\phi{}_t = \frac{\Omega_P}{8\pi r_0} \left( \frac{\beta_{S_0}}{2} + \beta_{D_0} + \frac{M}{r_0 \beta_{S_0}} + \frac{8\pi \rho r_0^2}{3\beta_{D_0}} \right) . \tag{27}$$

In order to relate the components of the energy-momentum tensor to the physical properties of the shell, we may use the decomposition (24) of the energy-momentum tensor. We need to find the vectors $u$ and $v_{(k)}$. The four-velocity of the points on the shell is

$$u^i = (1, 0, \omega_S) / \sqrt{{\beta_{S_0}}^2 - r_0^2 \sin^2(\omega_S - \Omega_P)^2} \, . \tag{28}$$

To first order in $\omega_S$ and $\Omega_P$ the four-velocity is

$$u^i = (1, 0, \omega_S) / \beta_{S_0} \, . \tag{29}$$

The two vectors $v_{(k)}$ must be chosen so that they are mutually orthogonal, normal, and orthogonal to $u$. Choosing

$$v^i_{(\theta)} = (r_0, 1/r_0, 0) \tag{30}$$

leads to

$$v^i_{(\phi)} = (r_0\beta_{S_0}{}^{-2}(\omega_S - \Omega_P)\sin\theta, 0, (r_0\sin\theta)^{-1}). \tag{31}$$

From eq. (24) we now find to first order in the angular velocities

$$S^t{}_t = \sigma, \tag{32}$$

$$S^\theta{}_\theta = S^\phi{}_\phi = -p, \tag{33}$$

$$S^\phi{}_t = \omega_S(\sigma + p). \tag{34}$$

Comparing these expressions with those in eqs. (25)–(27) we obtain

$$\sigma = \frac{\beta_{D_0} - \beta_{S_0}}{4\pi r_0}, \tag{35}$$

$$p = \frac{1}{8\pi r_0}\left(\frac{M}{r_0\beta_{S_0}} + \frac{8\pi\rho r_0^2}{3\beta_{D_0}}\right) - \frac{\sigma}{2}, \tag{36}$$

$$\frac{\Omega_P}{\omega_S} = \frac{8\pi(\sigma+p)}{4\pi(\sigma+2p) + \beta_{S_0}/2r_0 + \beta_{D_0}/r_0}, \tag{37}$$

or equivalently

$$\frac{\Omega_P}{\omega_S} = \frac{\beta_{S_0} - \beta_{D_0}(1 - 3M/r_0)}{\beta_{S_0} - \beta_{D_0}/2}. \tag{38}$$

It may be noted that there is no inertial dragging inside a spherical domain wall, which has $p = -\sigma$.

## 5. DISCUSSION

We shall first consider some special cases. If we put the shell at the position

$$r_0 = \left(\frac{3M}{8\pi\rho}\right)^{1/3} \tag{39}$$

then

$$\beta_{S_0} = \beta_{D_0} = \sqrt{1 - M/r_0} \tag{40}$$

and the components of the metric tensor are continuous across the shell. Inserting this into eqs. (29), (37) and (38) we get

$$\sigma = 0, \qquad p = \frac{M}{4\pi r_0^2\beta_0}, \qquad \frac{\Omega_P}{\omega_S} = \frac{2M}{r_0}. \tag{41}$$

This situation must be considered rather unphysical since the rest mass density of the shell vanishes.

Let us now compare our result with that of Brill and Cohen [3]. They considered a rotating shell with vanishing energy density in its interior. Inserting $\rho = 0$ and $\beta_{D_0} - 1$ into eq. (38) gives

$$\frac{\Omega_P}{\omega_S} = \frac{\beta_{S_0} + 3M/r_0 - 1}{\beta_{S_0} + 1/2}, \tag{42}$$

which looks different from the result of Brill and Cohen, eq. (5). However, our result is expressed in Schwarzschild coordinates while eq. (5) is valid in isotropic coordinates. The isotropic radial coordinate is related to the Schwarzschild coordinate by the transformation

$$r = \bar{r}(1 + M/2\bar{r})^2. \tag{43}$$

Using this in eq. (42), eq. (5) is recovered.

We shall now discuss the conditions for 'perfect dragging'. In the case considered by Brill and Cohen eqs. (35) and (36) reduce to

$$\sigma = \frac{1 - \beta_{S_0}}{4\pi r_0}, \qquad p = \frac{\beta_{S_0} + M/r_0 - 1}{8\pi r_0 \beta_{S_0}}. \tag{44}$$

Perfect dragging inside the shell happens for $r_0 = 2M$, i.e. $\beta_{S_0} = 0$, which leads to $\sigma = 1/8\pi M$, $p = \infty$.

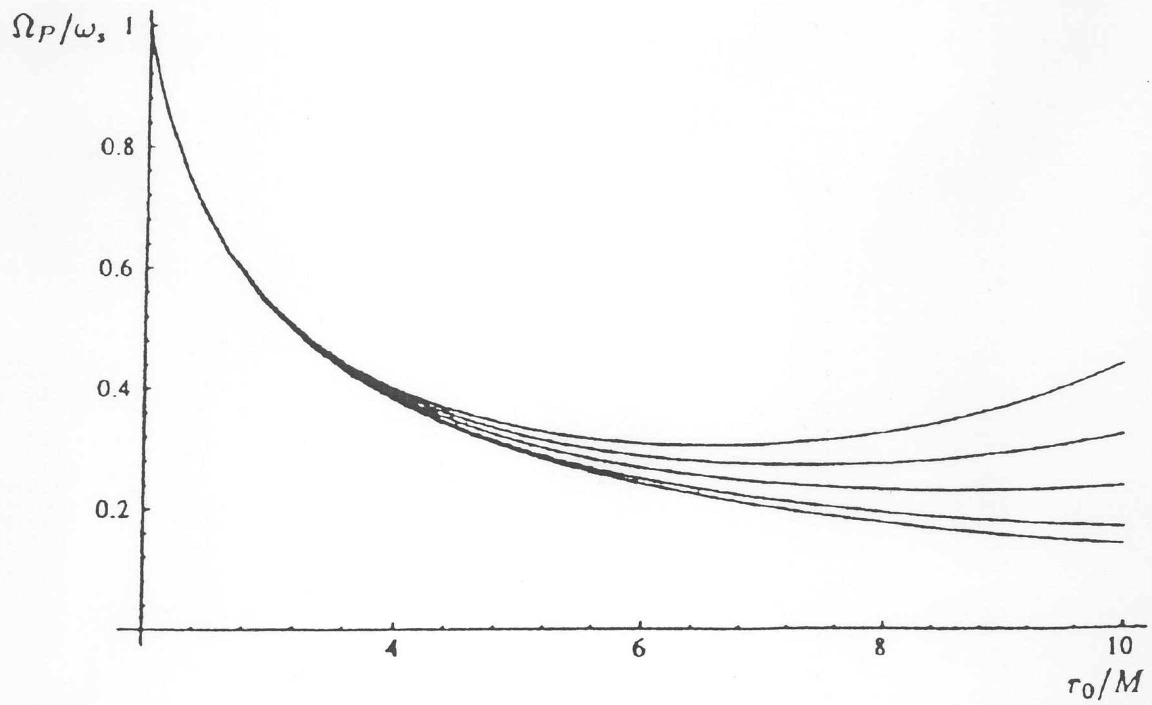


**Figure 1.** Angular dragging velocity in the interior of the shell as function of the radius of the shell for different values of the vacuum energy density. The lowest curve corresponds to $\rho = 0$.

In the general case, with non-vanishing vacuum energy inside the shell, we reformulate eq. (38) for the inertial dragging angular velocity as follows:

$$\frac{\Omega_P}{\omega_S} = 1 - \frac{3\beta_{D_0}{\beta_{S_0}}^2}{2\beta_{S_0} + \beta_{D_0}} . \tag{45}$$

This expression shows that for $M \geq 0$ and $\rho \geq 0$ we must have $\Omega_P \leq \omega_S$, i.e. we cannot have 'over-perfect' dragging.

There are now two possibilities that lead to perfect dragging: either $\beta_{S_0} = 0$, i.e. $r_0 = 2M$ which leads to $\sigma = \beta_{D_0} 4\pi r_0$, $p = \infty$, or $\beta_{D_0} = 0$, i.e. $r_0 = \sqrt{3/8\pi\rho}$ which leads to $\sigma = -\beta_{S_0}/4\pi r_0$, $p = \infty$. Both with and without vacuum energy inside the shell, perfect dragging only happens when the stresses in the shell diverge. Thus perfect dragging only takes place in a physically unobtainable limiting case.

In Figure 1 we have plotted the relative dragging angular velocity in the interior of the shell for different values of the energy density. We see that the dragging angular velocity increases with increasing vacuum energy density, indicating that this energy contributes to the dragging.

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