Vacuum energy and inertial dragging

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Preface

I would very much like to thank professor Øyvind Grøn for accepting me as his student, and for guiding me through this interesting four month project in a superb way.

Some of the calculations presented in this thesis would have been very heavy and time consuming to carry out if it was not for Harald H. Soleng's splendid macro package CARTAN for *Mathematica*. I would also like to thank the group for theoretical physics at UiO for letting me in.

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Introduction

When writing a thesis like this, one should know for certain what the reader is like. I have assumed that my reader knows much about classical mechanics, general relativity, and hence also differential geometry. Therefore, I shall not give any introduction to such topics here. In the main chapter, the one on rotating shells, we will need Israel's formalism for thin surfaces. I do not expect this to be a very well-known formalism, so I have included a short introduction here in order to make the thesis as self-contained as possible.

Conventions

Through this thesis I have tried to use a notation which does not differ too much from the one usually encountered in the litterature on general relativity. However, some points should be noted:

1. The signature is $(+ - --)$, *i.e.* time-like vectors have positive squared 'length,' while that of the space-like ones is negative.

2. Greek indeces such as μ, ν and λ are used to denote the components of tensors with respect to a *coordinate* frame, *e.g.* (t, x, y, z).

3. Components with respect to an orthonormal basis field are denoted with a hat, *e.g.* $\hat{\mu}$, taking the values $(0, 1, 2, 3)$, where the first one is time-like. The components of the metric tensor in a local orthonormal frame are denoted $\eta_{\hat{\mu}\hat{\nu}} = \mathrm{diag}(1, -1, -1, -1)$.

4. Vectors and tensors as geometrical objects are usually denoted with latin letters set in boldface, *e.g.* $\boldsymbol{a}$, while forms are denoted with boldface Greek letters, such as $\boldsymbol{\xi}$. The basis vectors are denoted $\boldsymbol{e}_\mu$, and the basis forms $\boldsymbol{\omega}^\mu$. One exception to this rule is that the operator giving the exterior derivative—which clearly is a form—is called $\boldsymbol{d}$.

Whenever any other notation is used, it will be explained.

We shall also be using geometrised units, *i.e.* $G = c = 1$, where G is Newton's gravitational constant, and c the speed of light. This means that everything will be measured in powers of the unit for lengths, which is taken to be metres (m).

1.1 Inertia and Mach's principle

The main goal for this thesis is to describe the connection between vacuum and inertia. Let us therefore see how the concept of inertia fits into the classical theories, and where the problems lie.

When Newton formulated his classical mechanics he was aware of the problems assosiated with inertia. In his theory some frames of reference were more "preferred" than others. These were called inertial frames, and in them the so-called inertial (or fictive) forces disappeared. Newton was very eager to find out the reason behind this selection of frames. First he observed that two inertial systems were moving with constant relative velocity. This is a variant of Newton's first law of motion. So, none of them were rotating nor accelerating relative to the others. It was then natural to ask what the origin of the inertial forces experienced in a rotating system was: the relative or the absolute (if any) motion? To find out of this Newton constructed his well-known experiment with the rotating vessel.

A vessel almost filled with water stays at rest relative to the earth. When having done so for a while, none of us is surprised to find that the surface of the water is flat. Then Newton makes the vessel start rotating. At first the water is unaffected; it is at rest relative to the earth, and the surface of the water remains flat. After a while, the water is rotating along with the vessel, and its surface becomes shaped like a paraboloid. Then Newton makes the vessel stop. At first the water rotates with a parabolically shaped surface, but after a while it stops, and becomes flat again. Newton's conclusion to this experiment is clear. The relative motion of the vessel and the water makes no contribution to the inertial forces in the water (which make its surface curve). It is the *absolute* motion of the water that matters. But, if it shall make any sense to speak of absolute motion, we must have an absolute space wherein it can take place. This was Newton's conclusion [1].

Many people have disliked this conclusion. Bishop Berkeley (1685–1753) commented that even though the water was at rest relative to the vessel, it was rotating relative to the earth and the fixed stars [2]. He meant that the effect of the vessel was negligible and that the effect was caused by the fixed stars.

Mach brought these ideas further [3]. He suggested that if the walls of the vessel were increased in thickness until they ultimately were several leagues thick, they would have the stronger impact on the water, which in turn would have a flat surface even though the vessel was rotating relative to the fixed stars. If this was so, he argued, the notion of "absolute space" is obsolete. This leads up to what is now known as *Mach's principle*, which can be formulated as

> *The inertial mass of a body is caused by its interactions with the other bodies in the universe.*

This surely reflects Mach's thoughts, but it was probably Einstein [4] who first formulated it as a *principle*, and referred to it as "Mach's principle."

The Sagnac experiment

We would like to have a device which is able to measure its rotation relative to an inertial frame—a sort of 'compass of inertia' (as required by Gödel [5]). We have already seen that a vessel filled with water can do the job, but it relies on the gravitational force of a nearby collection of matter (such as the earth). The famous Foucault pendulum also does the job, but it has the same restriction as the vessel.

Sagnac developed a device which was able to do the job: An optical fibre, or anything capable of carrying a photon in a closed curve, is arranged in a circle with radius r_0. At one point photons are emitted in both directions in the closed ring. After a while the photons return. In this way it is possible to measure the amount of time used by the photons around the circle. If they return simultaneously, the device is non-rotating, whereas it is rotating if they return at different instants. The difference in time consumption measures how fast the device is rotating relative to an inertial frame.

Let us carry this out in detail in an otherwise empty universe. The geometry is Minkowskian, given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2). \tag{1.1}$$

If the device is rotating relative to this system, it is at rest relative to another system given by $\varphi \mapsto \varphi - \omega t$, where ω is the angular velocity of the system. This gives the line element

$$\begin{aligned}\mathrm{d}s^2 &= \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta(\mathrm{d}\varphi - \omega\,\mathrm{d}t)^2\right)\\ &= (1 - r^2\omega^2\sin^2\vartheta)\mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) + 2r^2\omega\sin^2\vartheta\,\mathrm{d}\varphi\,\mathrm{d}t\end{aligned} \tag{1.2}$$

The circular ring is at rest in this system, with center at origo, and has radius r_0. The equation of motion for light in its interior is given by $\mathrm{d}s = \mathrm{d}r = \mathrm{d}\vartheta = 0$, or

$$0 = (1 - r_0^2\omega^2)\mathrm{d}t^2 - r_0^2\mathrm{d}\varphi^2 + 2r_0^2\omega\,\mathrm{d}\varphi\,\mathrm{d}t \tag{1.3}$$

$$\Rightarrow \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega \pm 1/r_0. \tag{1.4}$$

The term $1/r_0$ clearly is the angular velocity of light in the case of a non-rotating circuit, and $1/r_0 \gg \omega$. Therefore, in the $+$-solution the light travels in positive

φ-direction, while $\mathrm{d}\varphi/\mathrm{d}t < 0$ in the $-$-solution. Thus the times consumed by the two oppositely directed photons are

$$2\pi = (\omega + 1/r_0)t_+ \quad \Rightarrow \quad t_+ = \frac{2\pi r_0}{1 + \omega r_0}\,, \tag{1.5}$$

$$2\pi = (\omega - 1/r_0)t_- \quad \Rightarrow \quad t_- = \frac{2\pi r_0}{1 - \omega r_0}\,. \tag{1.6}$$

Hence, the time difference measured by our device is

$$\Delta t = t_- - t_+ = \frac{4\pi r_0^2 \omega}{1 - \omega^2 r_0^2}\,. \tag{1.7}$$

We notice that $\omega \to 0 \Rightarrow \Delta t \to 0$ and $r_0\omega \to 1 \Rightarrow \Delta t \to \infty$. This shows that the time difference measured by the Sagnac device is determined by its angular motion. When $\Delta t = 0$, the device is positioned in a non-rotating system. Thus, the Sagnac device is exactly what we wanted: a device which by measurements can separate rotating from non-rotating systems without having to rely on the gravitational forces of a nearby mass.

1.2 Dragging of inertial frames

When Mach presented his view on the rotating vessel, there were no theory which was capable of explaining the effects he predicted. The only known theory of gravitation was that of Newton, which certanly contained some weaknesses—*e.g.* that two bodies at different positions may act upon each other instantaneously. When Einstein presented his general theory of relativity, Mach's principle got its renaissance. According to this theory, inertia is a manifestation of the geometry of space–time. It also states that the geometry is affected by the presence of matter to an extent proportional to the factor G/c^2.

We may now consider the rotating vessel again. As a model for it we shall use a rotating sperical shell, with angular velocity ω_s, mass m_s and radius r_s. From dimensional analysis it is easy to deduce that the angular velocity of a nearby 'compass of inertia' is

$$\omega_d = k\frac{G}{c^2}\frac{m_s}{r_s}\omega_s, \tag{1.8}$$

where k is a numerical constant which has to be found by detailed calculations and d suggests 'dragging.' The angular dragging velocity may be measured by a Foucault pendulum or a Sagnac circuit. Lense and Thirring (1918) [6, 7] showed that

$$k = 4/3 \tag{1.9}$$

in the interior and at the poles of the spherical shell. However, they used a weak field approximation, and assumed slow rotation. Using geometrised units, the result may be written

$$\omega_d = \frac{4m_s\omega_s}{3r_s}. \tag{1.10}$$

The earth itself may be seen as a rotating vessel like this. Its dragging velocity is very small:

$$\omega_s \sim \frac{M_{\rm earth}\,\omega_{\rm earth}}{R_{\rm earth}} \sim 5 \cdot 10^{-14}\,\text{rad/s}, \tag{1.11}$$

and cannot be measured.

If we carry this calculation out for a pulsar with mass equal to that of the sun and a radius of 10 km performing 30 rotations per second, we get

$$\omega_d = \frac{M_{\rm pulsar}\,\omega_{\rm pulsar}}{R_{\rm pulsar}} \sim 0.15\,\omega_{\rm pulsar} \sim 30\,\text{rad/s} \sim 4.5\,\text{rotations/s}, \tag{1.12}$$

which is measurable indeed. However, the gravitational field surrounding a pulsar can hardly be regarded as weak.

The total dragging effects measured by a Foucault pendulum or a Sagnac device will in general depend on the combined effect of all masses in the universe. Hence, equation (1.10) more appropriately, if also somewhat symbolically, reads

$$\omega_d \sim \frac{m_s\omega_s}{r_s} + \sum_{\text{distant ``stars''}} \frac{M_{\text{``star''}}\,\omega_{\text{``star''}}}{R_{\text{``star''}}}. \tag{1.13}$$

Experience tells us that if there is no nearby star (or spherical shell), a 'compass of inertia' will not measure any rotation relatively to the stars. In this case we must have

$$\sum_{\text{``stars''}} \frac{M_{\text{``star''}}}{R_{\text{``star''}}} \sim \frac{M_{\rm universe}}{R_{\rm universe}} \sim 1. \tag{1.14}$$

This is exactly the relation between mass and radius of the universe at the maximum expansion of the Friedman model universes, which shows that the locally experienced inertial frames might be a consequence of the movement of the distant stars.

The locally measured inertia of a body seems to be determined by the existence of distant stars. How does the Minkowskian space–time fit into this picture? This manifold does certainly describe an *empty* space–time, but it is easy to see that test particles do have an inertia associated with them. This should be impossible according to Mach's principle. The Minkowskian space–time must therefore be seen as a limit space for closed space–times, which, according to Einstein, is what the general theory of relativity describes (The theory includes not only the geometrodynamic law, but also, in Einstein's view, the boundary condition that the universe be closed). The Minkowskian manifold may also be taken as a description of the space–time in the interior of a cosmic massive shell. Then there is no problem with the inertia associated with test particles.

In 1966 the result was generalised to strong fields by Brill & Cohen. Later we shall have a detailed look at their work, so we leave it for now.

1.3 This thesis

The next chapter will show us that vacuum is capable of containing energy, and that it, under certain circumstances, does. Since, again according to Einstein, mass and energy are equivalent, this means that the vacuum must exhibit some of the properties usually assosiated with "ordinary masses." What interests us is whether the vacuum has any inertial properties, whether it is capable of *moving*, and whether it has dragging properties like those mentioned above. The remaining part of the chapter will be concerned with general aspects of phenomena like acceleration and how to measure it.

In chapter 3 we shall try to describe how energetic vacuum looks to a linearly accelerated observer in order to find out something about the inertial properties of vacuum. The results are not promising to the wanted degree. In chapter 4 we shall revisit Newton's rotating vessel in form of a sperical shell, capable of containing energetic vacuum. This gives an interesting result, and in the last section we shall try to draw a conclusion to the question 'does the vacuum rotate?'

Chapter 2

Preliminaries

This chapter contains known results which will turn out to be a necessary background to the following chapters. They are included in order to make the thesis as self-contained as possible.

2.1 Polarized vacuum

From relativistic quantum mechanics we know that vacuum is not as empty as previously thought. There are always ongoing prosesses, such as creation and annihilation of elementary particles. Zel'dovich showed in 1968 [8] that this *vacuum polarization* gives rise to an energy–momentum tensor of the form

$$T_{\mu\nu} = \rho g_{\mu\nu}, \tag{2.1}$$

where the energy density ρ is positive.

In 1986 Grøn [9] showed, using a purely classical argument, that this form on the energy–momentum tensor follows from the Lorentz invariance of vacuum alone. Consider the components of the energy–momentum tensor $T_{\hat{\mu}\hat{\nu}}$ in a local orthonormal frame. According to any known experiment these values should be preserved under a Lorentz transformation (constant velocities are undetectable),

$$T_{\hat{\mu}\hat{\nu}} = T_{\hat{\mu}'\hat{\nu}'} = \Lambda^{\hat{\alpha}}_{\hat{\mu}} \Lambda^{\hat{\beta}}_{\hat{\nu}} T_{\hat{\alpha}\hat{\beta}}. \tag{2.2}$$

This relation must be satisfied for *any* Lorentz transformation, *e.g.* for conventional Lorentz boosts in the directions given by the coordinate axes. In the x^1-direction, the transformation matrix is

$$\Lambda^{\hat{\mu}}_{\hat{\mu}'} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \left(1 - v^2\right)^{-1/2}. \tag{2.3}$$

Performing Lorentz boosts in the three directions, and demanding invariance, gives

$$T_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} T_{00} & 0 & 0 & 0 \\ 0 & -T_{00} & 0 & 0 \\ 0 & 0 & -T_{00} & 0 \\ 0 & 0 & 0 & -T_{00} \end{pmatrix} = T_{00}\eta_{\hat{\mu}\hat{\nu}}. \tag{2.4}$$

Remembering that $T_{00} = T^{00}$ in a local orthonormal basis, we understand that T_{00} equals the proper energy density ρ of the vacuum. In homogeneous cosmological models the energy density as measured locally by an observer can be a function of time only. The relativity of simultaneity then gives that $\rho = \text{constant} = \rho_0$. Even though this is so, we shall usually be referring to it as ρ. Transforming the result to an arbitary basis $\{\boldsymbol{e}_\mu\}$ gives

$$T_{\mu\nu} = \rho g_{\mu\nu} \tag{2.5}$$

—the result obtained from quantum mechanics by Zel'dovich.

Vacuum as a perfect fluid

It is interesting to find that the above result for the energy–momentum tensor of vacuum fits into the formalism for perfect fluids.

Consider a portion of a perfect fluid. The fluid is taken to be continous, and is allowed to move freely. Insert an orthonormal comoving basis into this fluid. According to the observer carrying this frame, the fluid is locally at rest, whence Pascal's law for perfect fluids is valid: The pressure p applied to a given portion of the fluid is transmitted equally in all spatial directions and is everywhere perpendicular to the surface on which it acts. Thus, the three dimensional stress tensor is given by

$$\sigma^{\hat{\imath}\hat{\jmath}} = p\,\delta^{\hat{\imath}\hat{\jmath}}, \tag{2.6}$$

where $\hat{\imath} = 1, 2, 3$ covers the spatial directions.

The energy–momentum tensor may everywhere be written symbolically as

$$T^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} T^{00} & T^{0\hat{\imath}} \\ T^{\hat{\imath}0} & \sigma^{\hat{\imath}\hat{\jmath}} \end{pmatrix}. \tag{2.7}$$

Because we are positioned in a local orthonormal frame, $T^{00} = \rho$, and because the frame is moving together with the fluid, the energy transport, $T^{0\hat{\imath}} = T^{\hat{\imath}0}$, vanishes. Thus, the energy–momentum tensor in this special frame is

$$T^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \rho & 0 \\ 0 & \sigma^{\hat{\imath}\hat{\jmath}} \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{2.8}$$

We now want to find the energy–momentum tensor in *any* frame. Remembering that two tensors are *equal* if and only if they have the same components in one special basis, it is easy to find the general expression if we introduce the vector four velocity, u^μ:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p g^{\mu\nu}. \tag{2.9}$$

In a local orthonormal comoving basis, $u^{\hat{\mu}} = (1,0,0,0)$ and $g^{\hat{\mu}\hat{\nu}} = \eta^{\hat{\mu}\hat{\nu}}$, so the expression is clearly correct. How does the previously obtained energy–momentum tensor for the polarized vacuum fit into this? Well, letting

$$p = -\rho, \tag{2.10}$$

we see that the two expressions are equal. This equation is therefore usually referred to as the equation of state for the polarized vacuum.

Vacuum energy vs. a cosmological constant

We may now construct Einstein's field equations with polarized vacuum as the only source:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} = 8\pi\rho\, g_{\mu\nu}. \tag{2.11}$$

We notice that they closely resemble the source-free field equations with a cosmological constant Λ (as proposed by Einstein in order to obtain solutions describing static universes)

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 0 \quad \Leftrightarrow \quad G_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{2.12}$$

Comparing the two equations, we see that they are identical, with the formal substitution $\Lambda \leftrightarrow 8\pi\rho$. We might thus say that the vacuum energy acts as a cosmological constant, and there will be no observable difference between effects caused by $8\pi\rho$ and Λ.

Every known source-free solution of Einstein's field equations with a cosmological constant $\Lambda > 0$ is also a solution of the equations with polarized vacuum as the only source. The most famous and useful solution with these properties is the *de Sitter solution*.

The de Sitter solution

In 1917 de Sitter [10] found a solution of Einstein's field equations with positive cosmological constant. Later Eddington [11] gave it the form most used nowadays,

$$\mathrm{d}s^2 = \left(1 - \frac{\Lambda r^2}{3}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \Lambda r^2/3} - r^2\mathrm{d}\Omega^2, \quad \mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\!\vartheta\,\mathrm{d}\varphi^2. \tag{2.13}$$

Substituting $r = \sin R$ gives the original version found by de Sitter, where the radial coordinate R is the radial distance as measured by standard measuring rods. In the next section we will see that an observer at rest in this geometry will experience a repulsive gravitational force, which suggests that the solution is suited to describe an expanding universe. It is indeed, for it may be written

$$ds^2 = dt^2 - e^{2\sqrt{\Lambda/3}\,t}\left(dr^2 + r^2 d\Omega^2\right). \tag{2.14}$$

In this isotropic coordinate system every point of constant spatial coordiantes represent geodesics, and may be considered to be "at rest." According to this view, the solution describes a universe which expands exponentially with time, *i.e.* the distance between two points of constant spatial coordinates expands exponentially.

In chapter four we shall use the first version of the de Sitter line element, with the formal substitution $\Lambda \mapsto 8\pi\rho$, because it is *static*. We shall use it to describe a region of polarized vacuum inside a massive shell.

We notice that the static form of the de Sitter universe has a horizon at $r_0 = \sqrt{3/\Lambda}$. The expansion is so fast that points separated by more than this amount will be causally disconnected because light is incapable of reaching from the source to the destination before the distance has increased too much.

The expanding de Sitter solution is used to describe the so-called *inflationary era*, the epoch short after the creation of the universe during which it was vacuum dominated and expanded enormously. One believes that this epoch started about 10^{-35} s after the creation (when the GUT symmetries broke), and lasted until 10^{-33} s. During this relatively short period, the physical distances of the universe had increased by a factor 10^{30}. For an introduction to these phenomena, see Grøn [9].

2.2 The hyperbolically accelerated system

Constant linear acceleration in Minkowskian space–time is described by the so-called hyperbolically accelerated system. In the next chapter we shall try to describe accelerated motion in a universe containing polarized vacuum. It is therefore natural to give a short presentation of the hyperbolically accelerated system in order to have something to compare with later.

Let the coordinates of the Minkowskian space–time be denoted (T, X, Y, Z), and consider a particle of constant linear acceleration—$a^\mu a_\mu = -g^2$—in this system. Together with the usual restrictions on four velocities and accelerations, this gives the following set of equations for the motion of the particle:

$$a^\mu a_\mu = -g^2, \tag{2.15}$$

$$a^\mu u_\mu = 0, \tag{2.16}$$

$$u^\mu u_\mu = -1. \tag{2.17}$$

Chosing X as the direction of motion, these equations may be solved for a parametrization of the path of the particle

$$1 + gX = \cosh(g\tau), \tag{2.18}$$
$$gT = \sinh(g\tau), \tag{2.19}$$

where τ is the proper time of the particle. The constants of integration are chosen to give $X(\tau = 0) = T(\tau = 0) = 0$.

We want to know how the world looks to this particle. For this reason one may construct a *coordinate* system, where the particle is always at rest at origo. For a deduction of this system, see *e.g.* Misner *& al* [12]. If the new coordinates are denoted (t, x, y, z), the transformation between the coordinate systems is

$$gT = (1 + gx)\sinh(gt), \tag{2.20}$$
$$1 + gX = (1 + gx)\cosh(gt), \tag{2.21}$$
$$Y = y, \tag{2.22}$$
$$Z = z, \tag{2.23}$$

where the coordinate time t has been chosen to equal the proper time of the accelerated particle, $t = \tau$. In these coordinates the line element is

$$\mathrm{d}s^2 = (1 + gx)^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2. \tag{2.24}$$

It is easy to see that a horizon resides at $x = -1/g$, *i.e.* 'behind' the accelerated particle. This horizon is of the same type as the de Sitter horizon, and is caused by the impossibility for light to travel from a point behind the horizon to the observer in origo.

It is convenient to make yet another coordinate transformation, $\tilde{t} = gt$, $\tilde{x} = x + 1/g$, which gives the standard form of the line element of the *hyperbolically accelerated system*,

$$\mathrm{d}s^2 = x^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2, \tag{2.25}$$

where the tildas have been dropped. In this system the horizon is positioned at $x = 0$.

Vacuum energy in the hyperbolically accelerated system

It would be interesting to see what the energy–momentum tensor for polarized vacuum looks like in an accelerated system. This can be obtained by applying the transformation matrix,

$$M_\mu{}^{\hat{\mu}} = \frac{\partial x^{\hat{\mu}}}{\partial x^\mu} = \begin{pmatrix} (1 + gx)\cosh(gt) & \sinh(gt) & 0 & 0 \\ (1 + gx)\sinh(gt) & \cosh(gt) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.26}$$

according to the transformation rule

$$T_{\mu\nu} = M_\mu{}^{\hat{\mu}} M_\nu{}^{\hat{\nu}} T_{\hat{\mu}\hat{\nu}} = M_\mu{}^{\hat{\mu}} M_\nu{}^{\hat{\nu}} \rho\, \eta_{\hat{\mu}\hat{\nu}}. \tag{2.27}$$

Now hatted indices denote components in the Minkowskian space–time, while the unhatted ones denote components with respect to the hyperbolically accelerated system. This gives

$$T_{\mu\nu} = \rho \begin{pmatrix} (1+gx)^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \rho\, g_{\mu\nu}, \tag{2.28}$$

where $g_{\mu\nu}$ represents the metric tensor of the hyperbolic system. This indicates that the form of the energy–momenetum tensor is invariant under transformations involving acceleration. One should not be surprised by this result, since every equation between tensors have to remain valid under general coordinate transformations.

2.3 Measurements

We shall need a clear conception of which quantities are measurable in general. Components of vectors and tensors in an arbitary frame are not so, since the coordinate system is a way of referring to or labelling the space–time manifold. However, every *physical* quantity must be measurable by an observer. An observer is measuring at a predefined point of space, and he makes his measurements with standard clocks and measuring rods, or equivalently with his tetrad consisting of orthonormal vectors. His four basis vectors $\boldsymbol{e}_{\hat{\mu}}$ satisfies

$$\boldsymbol{e}_{\hat{\mu}} \cdot \boldsymbol{e}_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}. \tag{2.29}$$

In every such local orthonormal basis field $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$, whence the energy–momentum tensor for polarized vacuum as measured by our friend reads

$$T_{\hat{\mu}\hat{\nu}} = \rho\, \eta_{\hat{\mu}\hat{\nu}}. \tag{2.30}$$

According to this relation no transport of vacuum energy is possible, since $T^{0\hat{\imath}} = T^{\hat{\imath}0}$ regardless of the motion of the observer. This is also indicated by the fact that u^μ disappears in the expression for $T^{\mu\nu}$ for a perfect fluid when the equation of state for the polarized vacuum, $p = -\rho$ is inserted. Later in this thesis we shall see that a moving vacuum still might be possible because of its dragging effects.

Measuring gravity

Consider a free particle instantaneously at rest in a static gravitational field. The geodesic equation decomposed in an orthonormal basis field reads

$$\ddot{x}^{\hat{\mu}} + \Gamma^{\hat{\mu}}_{\hat{\kappa}\hat{\lambda}}\dot{x}^{\hat{\kappa}}\dot{x}^{\hat{\lambda}} = 0, \tag{2.31}$$

where the dot denotes differentiation with respect to the proper time. Remembering that the particle is at rest—$x^{\hat{\mu}} = (1, 0, 0, 0)$—we see that the three acceleration due to gravity as measured by standard clocks and rods is

$$\ddot{x}^{\hat{\imath}} = -\Gamma^{\hat{\imath}}_{00}. \tag{2.32}$$

In order to obtain an expression that does not diverge at the horizon of a black hole, the *acceleration of gravity* is definded to be

$$\kappa^i = \sqrt{g_{tt}}\,\ddot{x}^{\hat{\imath}} = -\sqrt{g_{tt}}\,\Gamma^{\hat{\imath}}_{00}. \tag{2.33}$$

Grøn has shown [13] that for a static spherically symmetric line element

$$\mathrm{d}s^2 = \mathrm{e}^{\nu(r)}\mathrm{d}t^2 - \mathrm{e}^{\lambda(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{2.34}$$

the gravitational acceleration is

$$\kappa \equiv \kappa^r = -\frac{1}{2}\mathrm{e}^{(\nu-\lambda)/2}\nu'. \tag{2.35}$$

If this is inserted into Einstein's field equations, one obtains

$$\kappa = -\frac{4\pi}{r^2}\int_0^r \left(T^0{}_0 - T^1{}_1 - T^2{}_2 - T^3{}_3\right)\mathrm{e}^{(\nu+\lambda)/2}\,r^2\,\mathrm{d}r \equiv -\frac{M_G}{r^2}. \tag{2.36}$$

In the Scwarzschild space–time this reduces to

$$\kappa = -\frac{M}{r^2}, \tag{2.37}$$

or, restoring the numerical factors,

$$\kappa = -\frac{GM}{r^2}, \tag{2.38}$$

where M is the gravitational or Schwarzschild mass of the system. This is exactly the same form as in Newtonian theory, except that the radial coordinate r differs from the physical radial distance.

Consider now a spherically symmetric system consisting of *polarized vacuum* with $T^{\mu}{}_{\nu} = \rho\,\delta^{\mu}_{\nu}$. Then

$$T^0{}_0 - T^1{}_1 - T^2{}_2 - T^3{}_3 = -2\rho < 0, \tag{2.39}$$

which shows that there is a gravitational repulsion away from the vacuum dominated region, as mentioned earlier.

An observer at rest in the *hyperbolically accelerated system* should also measure a gravitational force, as he is uniformly accelerated. Using the most physical coordinates where

$$\mathrm{d}s^2 = (1+gx)^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2, \tag{2.40}$$

we end up at

$$\kappa^x = -g \tag{2.41}$$

everywhere—as expected.

However, one question has to be asked: How does the last result agree with Mach's principle? It looks like inertial or gravitational forces may appear in general relativity without existence of anything else than a test particle. According to Mach's principle we are forced to take the view that Minkowskian space–time appears as a limit for closed universes, and that large masses exist indeed, even though they are 'infinitely far away.'

Chapter 3

Linear acceleration

Mach's principle states that the inertial forces are caused by gravitational interactions with the total mass of the universe. A usual way to describe his principle is "mass/energy *there* gives rise to inertia *here*." We have already seen that there may indeed be an energy assosiated with vacuum. It is then natural to ask wheather we may ascribe any Machian effects to vacuum.

We shall therefore try to describe vacuum with positive energy density, ρ, as seen by an accelerated observer. In order to rule out the possibility that the gravitational forces experienced by our observer are due to moving matter, it is nessecary to have vacuum *everywhere*. The inertial forces experienced must then be gravitational forces due to the motion of the vacuum relative to the observer. It is therefore interesting to find out whether such forces exist, and what they eventually look like. We thus want to find a solution of Einstein's field equations which can describe accelerated motion in a global vacuum, and which in the limit $\rho \to 0$ reduces to the well-known metric of a hyperbolically accelerated observer in Minkowskian space–time:

$$\mathrm{d}s^2 = x^2 \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2. \tag{3.1}$$

If a solution like this exists, we would like to argue that it describes motion in an infinitely extended vacuum, and that the forces experienced are due the relative motion of the vacuum. It will then be the analogue to the hyperbolic metric in a vacuum dominated space–time.

We have already seen that the components of the energy–momentum tensor must have the form $T_{\mu\nu} = \rho\, g_{\mu\nu}$ (we assume a vacuum dominated universe, as in the inflationary era), where ρ is a scalar function. In a local orthonormal frame this gives $T^{\hat{\mu}\hat{\nu}} = \rho\, \eta^{\hat{\mu}\hat{\nu}}$, and $T^{00} = \rho$. Therefore ρ is the energy density as measured by a local observer. The field equations of vacuum may then be written

$$G^{\mu}{}_{\nu} = 8\pi\rho\, \delta^{\mu}_{\nu}. \tag{3.2}$$

Before we start solving the equations we will explore the symmetries of the problem in order to find out something about the form of the line element.

3.1 Plane symmetric line element

We shall find the properties of space–time as measured by an observer which is moving through the vacuum with constant rest acceleration, *i.e.* $a^\mu a_\mu = -g^2$. For simplicity we assume that the motion is directed along the x-axis. We shall consider a space which admits plane symmetry, with the (y, z)-plane as symmetry plane. The most general line element for a space like this is given by Kramer, Stephani and Herlt [14]:

$$\mathrm{d}s^2 = f(x,t)\mathrm{d}t^2 - g(x,t)(\mathrm{d}y^2 + \mathrm{d}z^2) - h(x,t)\mathrm{d}x^2. \tag{3.3}$$

If the vacuum itself is static, there will be no time dependence in the metric. Our line element then takes the form

$$\mathrm{d}s^2 = f(x)\mathrm{d}t^2 - g(x)(\mathrm{d}y^2 + \mathrm{d}z^2) - h(x)\mathrm{d}x^2. \tag{3.4}$$

The coordinate transformation $\mathrm{d}\tilde{x} = \sqrt{h(x)}\,\mathrm{d}x$ removes $h(x)$ from the line element, and after renaming $\tilde{x} \mapsto x$, $f(x(\tilde{x})) \mapsto f(x)$ *&c*, we arrive at

$$\mathrm{d}s^2 = f(x)\mathrm{d}t^2 - g(x)(\mathrm{d}y^2 + \mathrm{d}z^2) - \mathrm{d}x^2, \tag{3.5}$$

with only two unknown functions to deal with.

It is tempting to remove the function $g(x)$ as well. That, however, is no good idea: the Einstein tensor corresponding to the resulting line element has two non-vanishing elements, namely $G^2{}_2$ and $G^3{}_3$. It is then impossible to solve $G^\mu{}_\nu = 8\pi\rho\,\delta^\mu_\nu$ unless $\rho = 0$; the case in which we are *not* interested... (If one tries to solve the field equations with $g(x) = 1$ and $\rho = 0$ one obtains the hyperbolic metric again.) Finally, we shall assume that the energy density of vacuum ρ is constant.

3.2 The field equations

It seems like we have to deal with the line element (3.5) as it stands. With respect to the local orthonormal frame

$$\boldsymbol{\omega}^0 = \sqrt{f(x)}\,\boldsymbol{d}t, \tag{3.6}$$

$$\boldsymbol{\omega}^1 = \boldsymbol{d}x, \tag{3.7}$$

$$\boldsymbol{\omega}^2 = \sqrt{g(x)}\,\boldsymbol{d}y, \tag{3.8}$$

$$\boldsymbol{\omega}^3 = \sqrt{g(x)}\,\boldsymbol{d}z, \tag{3.9}$$

the Einstein tensor has the non-vanishing components (computed by Cartan)

$$G^0{}_0 = \frac{g'^2 - 4gg''}{4g^2}, \tag{3.10}$$

$$G^1{}_1 = -\frac{g'\,(2gf' + fg')}{fg^2}\,, \tag{3.11}$$

$$G^2{}_2 = G^3{}_3 = \frac{g^2f'^2 + g'^2f^2 - fgf'g' - 2fg^2f'' - 2gf^2g''}{4f^2g^2}\,, \tag{3.12}$$

where the prime denotes differentiation whith respect to x.

The first of these gives rise to the equation for $g(x)$: $G^0{}_0 = 8\pi\rho$, or

$$g'^2 - 4gg'' = 32\pi\rho\, g^2. \tag{3.13}$$

Finding $g(x)$

It is fascinating that this equation can be solved exactly for $g(x)$. For, if we agree that $g(x) > 0$ (in order to preserve the signature of the metric and to avoid singularities), and assume that $g'(x) \neq 0$, we can multiply equation (3.13) by $3g^{-3/2}g'/2$:

$$-3\left(2g^{-1/2}g'g'' - \frac{1}{2}g^{-3/2}g'^3\right) = 32\pi\rho\frac{3}{2}g^{1/2}g', \tag{3.14}$$

which can be written

$$-3\frac{\mathrm{d}}{\mathrm{d}x}\left(g^{-1/2}g'^2\right) = 32\pi\rho\,\frac{\mathrm{d}}{\mathrm{d}x}\left(g^{3/2}\right). \tag{3.15}$$

We may now integrate, and if we call the constant of integration $4c_1$, we get

$$4c_1 - 3g^{-1/2}g'^2 = 32\pi\rho\, g^{3/2}, \tag{3.16}$$

alternatively

$$\sqrt{\frac{4c_1}{3}}\,\mathrm{d}x = \pm\frac{g^{-1/4}\,\mathrm{d}g}{\sqrt{1 - 8\pi\rho\, g^{3/2}/c_1}}\,. \tag{3.17}$$

Introducing a new variable $u = \sqrt{8\pi\rho/c_1}\, g^{3/4}$ this equation takes the form

$$\sqrt{6\pi\rho}\,\mathrm{d}x = \pm\frac{\mathrm{d}u}{\sqrt{1-u^2}}\,. \tag{3.18}$$

Integration gives

$$x - c_2 = \pm\frac{1}{\sqrt{6\pi\rho}}\arcsin\left(\sqrt{\frac{8\pi\rho}{c_1}}\,g^{3/4}\right), \tag{3.19}$$

which can be inverted to

$$g(x) = \left(\frac{c_1}{8\pi\rho}\right)^{2/3}\sin^{4/3}\left(\sqrt{6\pi\rho}\,(x - c_2)\right). \tag{3.20}$$

(The positive solution has been chosen in agreement with the assumption that $g(x) > 0$.)

We now have to choose the constants c_1 and c_2 such that this solution tends to the metric of hyperbolic motion in the limit $\rho \to 0$. We must therefore demand that $\lim_{\rho\to 0} g(x) = 1$. If we choose $c_2 = -\pi/(2\sqrt{6\pi\rho})$, *i.e.* $g(x) = (c_1/8\pi\rho)^{2/3}\cos^{4/3}\left(\sqrt{6\pi\rho}\,x\right)$, we have $\lim_{\rho\to 0} g(x) = \lim_{\rho\to 0}\sqrt{c_1/8\pi\rho}$, since $\cos x \to 1$ as $x \to 0$. We can now choose $c_1 = 8\pi\rho$, and obtain

$$\lim_{\rho\to 0} g(x) = 1. \tag{3.21}$$

Hence we have found the desired solution for $g(x)$:

$$g(x) = \cos^{4/3}\left(\sqrt{6\pi\rho}\,x\right). \tag{3.22}$$

Finding $f(x)$

Having found $g(x)$ we can use the $G^1{}_1$-component of the Einstein tensor to obtain an expression for $f(x)$.

$$8\pi\rho = G^1{}_1 = -\frac{g'\left(2gf' + fg'\right)}{fg^2} \tag{3.23}$$

$$\frac{f'}{f} = -\frac{g'^2 + 32\pi\rho\, g^2}{2gg'} \tag{3.24}$$

Substituting equation (3.13) into (3.24) leads to

$$\frac{f'}{f} = \frac{2g''}{g} - \frac{g'}{g}. \tag{3.25}$$

Integrations gives

$$\ln f = \ln c_3 + 2\ln g' - \ln g = \ln\left(\frac{c_3 g'^2}{g}\right) \tag{3.26}$$

$$\Rightarrow \quad f = \frac{c_3 g'^2}{g}, \tag{3.27}$$

where c_3 is a constant of integration. Inserting the solution (3.22) results in

$$f(x) = c_3\cos^{-2/3}\left(\sqrt{6\pi\rho}\,x\right)\sin^2\left(\sqrt{6\pi\rho}\,x\right). \tag{3.28}$$

As $\rho \to 0$, $\cos\left(\sqrt{6\pi\rho}\,x\right) \to 1$ and $\sin\left(\sqrt{6\pi\rho}\,x\right) \to \sqrt{6\pi\rho}\,x$, so $f(x) \to 6\pi\rho\, c_3 x^2$. If we choose $c_3 = 1/(6\pi\rho)$, $f(x) \to x^2$ as $\rho \to 0$, and we have found the desired solution:

$$f(x) = \frac{1}{6\pi\rho}\cos^{-2/3}\left(\sqrt{6\pi\rho}\,x\right)\sin^2\left(\sqrt{6\pi\rho}\,x\right). \tag{3.29}$$

which is valid for $0 < \sqrt{6\pi\rho}\,x < \pi/2$.

3.3 Discussion

We have now found a solution of the field equations of vacuum (3.2), which reduces to the hyperbolic metric as $\rho \to 0$. The solution can be written

$$\begin{aligned} \mathrm{d}s^2 = {} & \frac{1}{6\pi\rho}\cos^{-2/3}\Big(\sqrt{6\pi\rho}\,x\Big)\sin^2\Big(\sqrt{6\pi\rho}\,x\Big)\,\mathrm{d}t^2 \\ & - \cos^{4/3}\Big(\sqrt{6\pi\rho}\,x\Big)\,(\mathrm{d}y^2+\mathrm{d}z^2) - \mathrm{d}x^2. \end{aligned} \tag{3.30}$$

It is valid for $0 < \sqrt{6\pi\rho}\,x < \pi/2$. It follows by verification that it is also a solution of the two remaining equations corresponding to $G^2{}_2$ and $G^3{}_3$.

Since we have found a solution of $G_{\mu\nu} = 8\pi\rho\, g_{\mu\nu}$ it is natural to ask whether we have obtained an alternate form of the de Sitter metric, which is a solution of $G_{\mu\nu} = \Lambda g_{\mu\nu}$. (In which case we must have $8\pi\rho = \Lambda$.) It turns out that this is not the case: Cartan tells us that the Weyl invariant corresponding to our soluion is

$$C^{\kappa\lambda\mu\nu}C_{\kappa\lambda\mu\nu} = \frac{256\,\pi^2\rho^2}{3\cos\Big(\sqrt{6\pi\rho}\,x\Big)}\,, \tag{3.31}$$

whereas the Weyl invariant of the de Sitter metric vanishes.

To find out whether the solution (3.30) really describes an observer with constant rest acceleration we can calculate the acceleration of gravity he measures. Following the procedures given earlier one obtains

$$\kappa^x = -\frac{1}{3}\left(\cos^{-4/3}\Big(\sqrt{6\pi\rho}\,x\Big) + 2\cos^{2/3}\Big(\sqrt{6\pi\rho}\,x\Big)\right), \tag{3.32}$$

$$\kappa^y = \kappa^z = 0. \tag{3.33}$$

Thus the measured acceleration of gravity is *not* constant through the space. This indicates either that our observer does not travel with constant rest acceleration (as we demanded), or that we don't operate in a global vacuum (an observer with constant rest acceleration in empty, global vacuum should certanly measure a constant acceleration according to the principle of equivalence).

It turns out that the last of these solutions is the right one. The solution has been mentioned before without deductions, and it has been pointed out by Novotný [15], Novotný and Horský [16] and Amundsen and Grøn [17] that it describes the spacetime outside a massive plane. Anyway we have given a new deduction of it from the supposed plane symmetry and non-vanishing energy density (acting as a cosmological constant).

If we, instead of choosing c_2 as we did, had chosen $c_2 = 0$, and $g(x) = \sin^{4/3}(\sqrt{6\pi\rho}\,x)$, we would obtain the alternate form of the line element

$$\begin{aligned} \mathrm{d}s^2 = {} & \frac{1}{6\pi\rho}\sin^{-2/3}\Big(\sqrt{6\pi\rho}\,x\Big)\cos^2\Big(\sqrt{6\pi\rho}\,x\Big)\,\mathrm{d}t^2 \\ & - \sin^{4/3}\Big(\sqrt{6\pi\rho}\,x\Big)\,(\mathrm{d}y^2+\mathrm{d}z^2) - \mathrm{d}x^2. \end{aligned} \tag{3.34}$$

This and the earlier result can both be derived from the generalized Taub solution (see Novotný [15])

$$\mathrm{d}s^2 = \frac{1 - 8\pi\rho\,\xi^3/3}{\xi}\,\mathrm{d}\eta^2 - \frac{\xi}{1 - 8\pi\rho\,\xi^3/3}\,\mathrm{d}\xi^2 - \xi^2(\mathrm{d}y^2 + \mathrm{d}z^2), \tag{3.35}$$

by the coordinate transformations

$$\xi = A\sin^{2/3}\Big(\sqrt{6\pi\rho}\,x\Big)\,, \tag{3.36}$$

$$\text{and}\quad \xi = A\cos^{2/3}\Big(\sqrt{6\pi\rho}\,x\Big)\,, \tag{3.37}$$

respectively. Here A is a well-chosen constant, and η is a rescaled version of t. Our solution describes the part of the generalized Taub solution where $0 < \xi < A$.

We should feel a great relief that we have approached a "wrong" solution. If this was the right one we would have been in serious trouble indeed. First of all the locally measured acceleration of gravity is not constant through the space. More serious is the resulting contradiction to Mach's principle: The observer measures gravity, while the energy–momentum tensor shows no signs of a moving vacuum since it is, by hypothesis, diagonal like the metric tensor. Physically this means that no energy transport is associated with the vacuum. The observer is then in a sense stationary with respect to it. Then, according to Mach, there should be no forces of inertia. But the observer experiences forces! If he was positioned in a space with vacuum everywhere these forces could not be of a gravitational nature; they would have to be inertial forces. The inertial forces would then exist without any interaction with the surrounding world, which is exacly what Mach's principle states as impossible.

We are thus saved by the massive plane. However it leaves another question for us: What went wrong? The mathematical deduction of the solution is clearly correct, so the error must appear in the guesses we made on the form of the line element. These were 'diagonal metric' and 'static vacuum.' Both seem to fail. If the metric is diagonal, there can be no transport of energy assosiated with the vacuum (since $T^{0i} = 0$). A more realistic attempt would be to introduce an additional term proportional to $\mathrm{d}x\,\mathrm{d}t$ in the line element. We have assumed a global *static* vacuum with non-vanishing energy density ρ. Since the energy density acts as a cosmological constant, our solution must be an accelerated version of an empty-space solution with a such constant present. These solutions describe *expanding* universes, and we understand that it was too optimistic to hope for a static solution.

Chapter 4

Rotating shells

In the previous chapter we tried to describe how a vacuum-dominated universe looked to an accelerated observer. The conclusion was that we failed. Linear acceleration seems difficult to work with. It is however a well-known fact that rotation as well as linear acceleration gives rise to inertial forces. So, instead of looking for an analogue to the hyperbolic metric in a vacuum-dominated universe, we might try to find an analogue to Newton's vessel.

General relativistic attempts have been made toward this before in the case where $\rho = 0$. We have already mentioned the result of Lense and Thirring. They considered space–time outside and inside a slowly rotating shell in the weak field approximation of the field equations. In 1966 Brill & Cohen [18] carried out a calculation of the same problem without assuming a weak gravitational field, and two years later they generalised their result to incompressible fluid spheres [19]. Here we shall go through their calculations, using a slightly different approach, since we shall need their techniques later.

Until now nobody has considered how polarized vacuum is affected by rotating matter. After reproducing the result of Brill & Cohen, we shall use the same strategy to try to find a solution when there is a constant positive energy density ρ and stress $p = -\rho$ in the interior of the shell. It turns out that we must then introduce some serious restrictions in order to be able to carry out the calculations. Afterwards we show how one may avoid this problem by means of Israel's formalism [20] for surface layers. In this way we obtain a general result, which can be checked out against the result of Brill & Cohen, the well-known result of Lense & Thirring, and the special case considered in section 2 of this chapter.

4.1 Brill & Cohen's rotating "vessel"

In their article of 1966 Brill & Cohen take a thin spherical shell as their model for Newton's vessel. If there is no vacuum energy outside the shell, the theorem of Birkhoff [21] tells us that we must have Schwarzschild space–time outside the shell when it is not rotating. Inside the shell space–time is Minkowskian. To obtain the simplest mathematical treatment of the problem, we shall adopt the use of *isotropic* coordinates for the Schwarzschild geometry. In these coordinates the Schwarzschild line element reads

$$\mathrm{d}s^2 = \left(\frac{\tilde{r} - M/2}{\tilde{r} + M/2}\right)\mathrm{d}t^2 - \left(1 + \frac{M}{2\tilde{r}}\right)^4 \left(\mathrm{d}\tilde{r}^2 + \tilde{r}^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\right), \tag{4.1}$$

where M is the gravitational mass of the shell. The isotropic radial coordinate $\tilde{r}$ is connected to the standard Schwarzschild radial coordinate via the transformation

$$r = \tilde{r}\left(1 - \frac{M}{2\tilde{r}}\right)^2 \quad \Leftrightarrow \quad 2\tilde{r} = \sqrt{r^2 - 2Mr} + r - M, \tag{4.2}$$

hence the event horizon at $r = 2M$ in standard Schwarzschild coordinates resides at $\tilde{r} = M/2$ in isotropic coordinates.

The other coordinates have the same meaning as the Schwarzschild ones. If the shell is positioned at $\tilde{r} = \tilde{r}_0$, and the geometry is flat (*i.e.* Minkowskian) in its interior, the line element can be written on the compact form

$$\mathrm{d}s^2 = V^2\mathrm{d}t^2 - \psi^4\left(\mathrm{d}\tilde{r}^2 + \tilde{r}^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\right), \tag{4.3}$$

with

$$V = \begin{cases} (\tilde{r} - \alpha)/(\tilde{r} + \alpha) & \text{for } \tilde{r} > \tilde{r}_0 \\ (\tilde{r}_0 - \alpha)/(\tilde{r}_0 + \alpha) \equiv V_0 & \text{for } \tilde{r} < \tilde{r}_0 \end{cases}, \tag{4.4}$$

$$\psi = \begin{cases} 1 + \alpha/\tilde{r} & \text{for } \tilde{r} > \tilde{r}_0 \\ 1 + \alpha/\tilde{r}_0 \equiv \psi_0 & \text{for } \tilde{r} < \tilde{r}_0 \end{cases}, \tag{4.5}$$

where the convenient notation $\alpha = M/2$ has been introduced. The constants V_0 and ψ_0 assure continuity of the metric tensor across the shell.

If the shell is not static, but slowly rotating with angular velocity ω_s (s suggests "shell") in the φ direction, the rotation can be included in the line element as a perturbation on the static metric (as suggested by Thirring's weak field result):

$$\mathrm{d}s^2 = V^2\mathrm{d}t^2 - \psi^4\left(\mathrm{d}\tilde{r}^2 + \tilde{r}^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta(\mathrm{d}\varphi - \Omega(\tilde{r})\mathrm{d}t)^2)\right), \tag{4.6}$$

with V and ψ unaltered. The function $\Omega(r)$ represents the dragging of the inertial frames which can be measured by a compass of inertia—exactly what we want to find. In the weak field approximation it must be constant inside the shell, as showed by Lense and Thirring. In order to avoid inconsistency with the static case, we demand that $\Omega = 0$ when $\omega_s = 0$. We should also always have $\Omega \leq \omega_s$, since the induced dragging velocity never can be faster than that of the dragging shell. All calculations will be carried out to first order in ω_s. Since $\Omega \leq \omega_s$, we will also neglect higher order terms in Ω. For simplicity we shall also assume that Ω varies slowly with $\tilde{r}$, and consequently neglect higher order terms in its derivatives as well.

A convenient local orthonormal frame is

$$\boldsymbol{\omega}^0 = V\boldsymbol{d}t, \tag{4.7}$$

$$\boldsymbol{\omega}^1 = \psi^2\boldsymbol{d}\tilde{r}, \tag{4.8}$$

$$\boldsymbol{\omega}^2 = \tilde{r}\psi^2\boldsymbol{d}\vartheta, \tag{4.9}$$

$$\boldsymbol{\omega}^3 = \tilde{r}\psi^2\sin\vartheta(\boldsymbol{d}\varphi - \Omega\boldsymbol{d}t). \tag{4.10}$$

In this frame the components of the Einstein tensor corresponding to the line element (4.3) (given by CARTAN) to first order in ω_s, Ω and its derivatives are

$$G^{00} = -\frac{4(2\psi' + \tilde{r}\psi'')}{\tilde{r}\psi^5}, \tag{4.11}$$

$$G^{11} = \frac{2(2\psi V\psi' + 2\tilde{r}V\psi'^2 + \psi^2 V' + 2\tilde{r}\psi\psi' V')}{\tilde{r}\psi^6 V}, \tag{4.12}$$

$$G^{22} = G^{33} = \frac{2\psi V\psi' - 2\tilde{r}V\psi'^2 + \psi^2 V' + 2\tilde{r}\psi V\psi'' + \tilde{r}\psi^2 V''}{\tilde{r}\psi^6 V}, \tag{4.13}$$

$$G^{03} = G^{30} = \frac{\sin\vartheta}{2\psi^3 V^2}(-4\psi V\Omega' - 6\tilde{r}V\Omega'\psi' + \tilde{r}\psi\Omega' V' - \tilde{r}\psi V\Omega''), \tag{4.14}$$

where the primes denote differentiation with respect to $\tilde{r}$. We notice that to first order in ω_s the diagonal components of the Einstein tensor are unaffected by the rotation. Into these expressions we must insert the known functions ψ, V and their derivatives, which are

$$\psi' = -\frac{\alpha}{\tilde{r}^2}\,\theta(\tilde{r} - \tilde{r}_0), \tag{4.15}$$

$$\psi'' = \frac{2\alpha}{\tilde{r}^3}\,\theta(\tilde{r} - \tilde{r}_0) - \frac{\alpha}{\tilde{r}^2}\,\delta(\tilde{r} - \tilde{r}_0), \tag{4.16}$$

$$V' = \frac{2\alpha}{(\tilde{r} + \alpha)^2}\,\theta(\tilde{r} - \tilde{r}_0), \tag{4.17}$$

$$V'' = -\frac{4\alpha}{(\tilde{r} + \alpha)^3}\,\theta(\tilde{r} - \tilde{r}_0) + \frac{2\alpha}{(\tilde{r} + \alpha)^2}\,\delta(\tilde{r} - \tilde{r}_0). \tag{4.18}$$

Here $\theta(r)$ is the step function, and $\delta(r)$ is the Dirac delta function.

Because the shell is positioned in a vacuum with vanishing energy density ρ, we know that $T^{\hat{\mu}\hat{\nu}} = 0$ everywhere, with a possible exception at the shell. We shall now integrate the field equations across the shell in order to find $T^{\hat{\mu}\hat{\nu}}$ there. Introduce the notation

$$\int A \equiv \lim_{\varepsilon\to 0}\int_{\tilde{r}_0-\varepsilon}^{\tilde{r}_0+\varepsilon} A(r)\,\mathrm{d}r. \tag{4.19}$$

When evaluating integrals like this, only the terms containing a delta function—which happens to be exactly those containing second derivatives of ψ and V—will contribute. We start with the 00-component of the energy–momentum tensor which must equal σ, the rest mass density of the shell, since we are operating in a local orthonormal frame:

$$8\pi \int T^{00} = \int G^{00} = -4 \int \frac{\psi''}{\psi^5} = 4\alpha \int \frac{\delta(r - r_0)}{r^2 \psi^5} = \frac{4\alpha}{\tilde{r}_0^2 \psi_0^5} . \tag{4.20}$$

Since $T^{\mu\nu} = 0$ everywhere except on the shell, this means that

$$\sigma = T^{00} = \frac{\alpha\, \delta(\tilde{r} - \tilde{r}_0)}{2\pi \tilde{r}^2 \psi^5} . \tag{4.21}$$

Taking a closer look at G^{11} we notice that it only contains first derivatives of ψ and V. Consequently, $T^{11} = 0$ at the shell as well as elsewhere, meaning that there are no radial stresses in the shell. Together with the tangential stresses, this causes the shell to be stationary.

Integrating G^{22} across the shell yields

$$\begin{aligned} 8\pi \int T^{22} &= \int G^{22} \\ &= \int \left(\frac{2\psi''}{\psi^2} + \frac{V''}{\psi^4 V} \right) \\ &= -\frac{2\alpha}{\tilde{r}_0^2 \psi_0^5} + \frac{2\alpha}{(\tilde{r}_0 + \alpha)^2 \psi_0^4 V_0} \\ &= \frac{\alpha}{2(\tilde{r}_0 - \alpha)} \frac{4\alpha}{\tilde{r}_0^2 \psi_0^5} \\ &= \frac{\alpha}{2(\tilde{r}_0 - \alpha)}\, 8\pi \int T^{00}, \end{aligned} \tag{4.22}$$

which shows that

$$T^{22} = T^{33} = \frac{\sigma\alpha}{2(\tilde{r}_0 - \alpha)} \equiv \sigma\beta. \tag{4.23}$$

We have now found the diagonal components of the energy–momentum tensor at the shell. To find a useful expression for the 03-component, we must first use the field equations outside and inside the shell to find out something about the induced rotation, $\Omega(\tilde{r})$.

The induced rotation

We have already stated that $T^{\hat{\mu}\hat{\nu}} = 0$ outside and inside the shell, so the remaining field equation reads $G^{03} = 0$. This equation can be integrated for $\Omega(\tilde{r})$. First we notice that G^{03} may be written as

$$G^{03} = -\frac{\sin\vartheta}{2\tilde{r}^3 \psi^8} \left(\frac{r^4 \psi^6 \Omega'}{V} \right)' \tag{4.24}$$

so that the corresponding field equation is

$$\left(\frac{(\tilde{r}\psi^2)^4\Omega'}{V\psi^2}\right)' = 0, \tag{4.25}$$

for $\tilde{r} \neq \tilde{r}_0$. Integration results in

$$\Omega' = \frac{KV\psi^2}{(\tilde{r}\psi^2)^4}, \tag{4.26}$$

where K is a constant of integration.

Inside the shell both ψ and V are constant, so $\Omega' \sim 1/r^4$. Hence $\Omega = A/r^3 + \Omega_B$, where A and Ω_B are constants (B suggests 'Brill & Cohen' as there shall be several versions of this constant later). However, the solution must be regular in origo, and we have to choose $A = 0$. The result—that $\Omega = \Omega_B$ inside the shell—closely resembles that of Lense and Thirring. It remains to see whether they are of same magnitude.

For $\tilde{r} > \tilde{r}_0$ we can insert the expressions for ψ and V and integrate. This results in

$$\Omega = -\frac{K}{3(\tilde{r}\psi^2)^3} + B, \tag{4.27}$$

where B is another constant of integration. If we *choose* $B = 0$, we obtain $\lim_{\tilde{r}\to\infty} \Omega(\tilde{r}) = 0$. This is a natural choice, since an observer at infintity then is non-rotating. Furthermore, $\Omega(\tilde{r})$ describes a local rotation of the inertial frames with respect to a non-rotating stationary observer at infinity.

Continuity across the shell requires

$$\Omega_B = -\frac{K}{3(\tilde{r}_0\psi_0^2)^3}. \tag{4.28}$$

Inserting this value of K into the expression for Ω gives

$$\Omega(\tilde{r}) = \begin{cases} \Omega_B \left(\dfrac{\tilde{r}_0\psi_0^2}{\tilde{r}\psi^2}\right)^3, & \tilde{r} > \tilde{r}_0. \\ \Omega_B, & \tilde{r} < \tilde{r}_0. \end{cases} \tag{4.29}$$

It remains to determine the value of Ω_B. It should indeed be related to the rotation of the shell, ω_s, in some way. To figure this out, we must integrate the remaining field equation across the shell. We will then need the first and second derivatives of $\Omega(\tilde{r})$:

$$\Omega' = -\frac{3\Omega_B(\tilde{r}_0\psi_0^2)^3\psi^2}{(r\psi^2)^4}\,\theta(\tilde{r} - \tilde{r}_0), \tag{4.30}$$

$$\Omega'' = -\left(\frac{3\Omega_B(\tilde{r}_0\psi_0^2)^3\psi^2}{(r\psi^2)^4}\right)'\theta(\tilde{r} - \tilde{r}_0) - \frac{3\Omega_B(\tilde{r}_0\psi_0^2)^3\psi^2}{(r\psi^2)^4}\,\delta(\tilde{r} - \tilde{r}_0). \tag{4.31}$$

Taking a look at G^{03} we see that only the delta function will contribute to T^{03} of the shell:

$$8\pi \int T^{03} = \int G^{03} = -\int \frac{r\psi V \Omega'' \sin\vartheta}{2\psi^3 V^2} = \frac{3\Omega_B \sin\vartheta}{2\psi_0^2}. \tag{4.32}$$

We now have the following information regarding the energy–momentum tensor for the shell:

$$T^{00} = \sigma, \tag{4.33}$$
$$T^{11} = 0, \tag{4.34}$$
$$T^{22} = T^{33} = \sigma\beta, \tag{4.35}$$
$$8\pi \int T^{03} = 8\pi \int T^{30} = \frac{3\Omega_B \sin\vartheta}{2\psi_0^2}. \tag{4.36}$$

The left hand side of this equation will now be calculated in terms of the angular velocity of the shell. If the shell consisted of dust, the energy–momentum tensor would be $T^{\mu\nu} = \sigma u^\mu u^\nu$, where σ is again the rest mass density, and u^μ is the velocity four vector. In our case the shell is capable of having stresses as well. The energy–momentum tensor in a local orthonormal frame must then have the form

$$T^{\hat\mu\hat\nu} = \sigma u^{\hat\mu} u^{\hat\nu} + \sum_{i,j=1}^{3} t^{ij} {v_{(i)}}^{\hat\mu} {v_{(j)}}^{\hat\nu}, \tag{4.37}$$

where the ${v_{(i)}}^{\hat\mu}$ form a triad of orthonormal vectors spanning the hypersurface orthogonal to $u^{\hat\mu}$. $T^{\hat\mu\hat\nu}$ must have this form because in the rest frame of the matter the momentum density T^{0i} should vanish. Also t^{ij} must be diagonal in order to make $T^{\hat\mu\hat\nu}$ symmetric about the equatorial plane and with respect to time reversal.

We shall now calculate the components of $u^{\hat\mu}$ in the local orthonormal frame given above, and then construct a set of vectors ${v_{(i)}}^{\hat\mu}$ with the wanted properties. In a coordinate frame the components of u^μ are given by $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. Transforming this to the local orthonormal frame (4.7–4.10) gives $u^{\hat\mu} = M^{\hat\mu}{}_\mu u^\mu$, where $\boldsymbol{\omega}^{\hat\mu} = M^{\hat\mu}{}_\mu \boldsymbol{d}x^\mu$. Hence the velocity four vector may be written symbolically as

$$u^{\hat\mu} = \frac{\omega^{\hat\mu}}{\mathrm{d}\tau} \tag{4.38}$$

in the local orthonormal frame. We have introduced the notation $\omega^{\hat\mu} = M^{\hat\mu}{}_\mu \mathrm{d}x^\mu$. For the shell we have $\mathrm{d}\tilde r = \mathrm{d}\vartheta = 0$, $\mathrm{d}\varphi/\mathrm{d}t = \omega_s$ and $\mathrm{d}\tau = \sqrt{V^2 \mathrm{d}t^2 - \psi^4 \tilde r^2 \sin^2\vartheta (\mathrm{d}\varphi - \Omega(r)\mathrm{d}t)^2}$, whence

$$u^0 = (1-\lambda^2)^{-1/2}, \tag{4.39}$$
$$u^1 = 0, \tag{4.40}$$
$$u^2 = 0, \tag{4.41}$$
$$u^3 = \lambda(1-\lambda^2)^{-1/2}, \tag{4.42}$$

with $\lambda = \tilde{r}\psi^2 \sin\vartheta(\omega_s - \Omega)/V$. Writing this as

$$u^{\hat{\mu}} = (1, 0, 0, \lambda)/\sqrt{1-\lambda^2}, \tag{4.43}$$

it is easy to see that the vectors $v_{(i)}{}^{\hat{\mu}}$ may be chosen as

$$v_{(1)}{}^{\hat{\mu}} = (0, 1, 0, 0), \tag{4.44}$$
$$v_{(2)}{}^{\hat{\mu}} = (0, 0, 1, 0), \tag{4.45}$$
$$v_{(3)}{}^{\hat{\mu}} = (\lambda, 0, 0, 1)/\sqrt{1-\lambda^2}\,. \tag{4.46}$$

Inserting these vectors into the general expression for the energy–momentum tensor, using that t^{ij} is diagonal gives, correct to first order in ω_s (and hence also in Ω, its derivatives and λ)

$$T^{00} = \sigma, \tag{4.47}$$
$$T^{11} = t^{11}, \tag{4.48}$$
$$T^{22} = t^{22}, \tag{4.49}$$
$$T^{33} = t^{33}, \tag{4.50}$$
$$T^{03} = T^{30} = (\sigma + t^{33})\lambda. \tag{4.51}$$

Comparing the expressions (4.47–4.50) with the previous result, equations (4.33–4.35), we see that

$$t^{11} = 0, \tag{4.52}$$
$$t^{22} = t^{33} = \sigma\beta. \tag{4.53}$$

This can be inserted into (4.51), which in turn can be integrated through the shell. This gives

$$8\pi \int T^{03} = 8\pi \int \sigma\lambda(1+\beta) = \frac{4\alpha}{\tilde{r}_0\psi_0^3 V_0}(1+\beta)(\omega_s - \Omega_B)\sin\vartheta, \tag{4.54}$$

where the expression for σ, equation (4.21), has been inserted. Finally, this result can be inserted into (4.36) and solved for Ω_B. This gives the result of Brill and Cohen:

$$\Omega_B = \frac{\omega_s}{1 + \dfrac{3(\tilde{r}_0 - \alpha)}{4M(1+\beta)}}\,, \tag{4.55}$$

which can also be writen

$$\frac{\Omega_B}{\omega_s} = \left(1 + \frac{3}{4M}\,\frac{(\tilde{r}_0 - M/2)^2}{\tilde{r}_0 - M/4}\right)^{-1}. \tag{4.56}$$

This result has a couple of interesting properties. First of all it reduces to the well-known result of Lense and Thirring if we restrict ourselves to the weak field limit. Formally, this can be done by letting $\alpha \to 0$, which gives

$$\Omega_B = \frac{4M\omega_s}{3\tilde{r}_0}. \tag{4.57}$$

We also observe that perfect dragging, *i.e.* $\Omega_B = \omega_s$—the induced rotation inside the shell equals the rotation of the shell—is possible. It occurs if the shell is positioned at its Schwarzschild radius, $\tilde{r}_0 = \alpha$. A shell of matter of radius equal to its Schwarzschild radius has been taken as an idealized cosmological model of our universe. This result shows that in such a model there cannot be a rotation of the local inertial frames relative to the large masses in the universe. This was pointed out by Brill and Cohen. However, in the limit $\tilde{r}_0 \to \alpha$ the shell becomes rather unphysical. As we shall see later using Israel's formalism, the stresses on the shell diverge, which indicates that the model has to be changed.

4.2 Rotating shell containing polarized vacuum

The previous section showed us how the local inertial frames are affected by a nearby rotating mass: there are indeed Machian effects associated with it. Since vacuum—as well as mass—may contain energy, we would like to find a similar dragging effect if the *vacuum* itself rotates.

Consider a spherical "ball" consisting of polarized vacuum with positive energy density ρ and stress $p = -\rho$. We want this situation to be stationary, whence the boundary of the "ball" has to exhibit certain properties. This boundary may be seen as a shell with polarized vacuum where $T_{\mu\nu} = \rho g_{\mu\nu}$ in its interior, and unpolarized vacuum with $T_{\mu\nu} = 0$ outside. According to Birkhoff's theorem the space–time outside a spherical symmetric constellation of mass or energy is of Schwarzschild type. Inside we should have the usual de Sitter space–time.

The shell or boundary is now assumed to rotate slowly, with angular velocity ω_s. As before the effect of this rotation is introduced as a perturbation $\Omega(r)$ of the static metric. The metric now reads

$$\mathrm{d}s^2 = f(r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{f(r)} - r^2\left(\mathrm{d}\vartheta^2 + \sin^2\!\vartheta(\mathrm{d}\varphi - \Omega(r)\mathrm{d}t)^2\right), \tag{4.58}$$

$$\text{where} \quad f(r) = \begin{cases} 1 - 2M/r & \text{for } r > r_0 \\ 1 - 8\pi\rho r^2/3 & \text{for } r < r_0 \end{cases}, \tag{4.59}$$

when the standard Schwarzschild and static de Sitter coordinates are used. We now want to carry out the same calculations as in the previous section for this situation. To make this possible, we must demand that the metric tensor is continous across the shell (this was automatically satisfied before). This requires the relation

$$r_0 = \left(\frac{3M}{4\pi\rho}\right)^{1/3} \tag{4.60}$$

to be satisfied. As we shall see later, this is an extremely special case—we are not allowed to put the shell anywhere. Let us carry out the calculations anyway.

As before we start with extracting as much information as possible out of the field equation in a local orthonormal frame. A convenient frame is

$$\boldsymbol{\omega}^0 = \sqrt{f(r)}\,\boldsymbol{d}t, \tag{4.61}$$

$$\boldsymbol{\omega}^1 = \frac{\boldsymbol{d}r}{\sqrt{f(r)}}, \tag{4.62}$$

$$\boldsymbol{\omega}^2 = r\,\boldsymbol{d}\vartheta, \tag{4.63}$$

$$\boldsymbol{\omega}^3 = r\sin\vartheta\,(\boldsymbol{d}\varphi - \Omega(r)\boldsymbol{d}t)\,. \tag{4.64}$$

In this frame the non-vanishing components of the Einstein tensor to first order in Ω and its derivatives are (again as given by CARTAN):

$$G^{00} = -G^{11} = \frac{1-f-rf'}{r^2}, \tag{4.65}$$

$$G^{22} = G^{33} = \frac{2f'+rf''}{2r}, \tag{4.66}$$

$$G^{03} = G^{30} = -\frac{1}{2}\left(\sqrt{f}\sin\vartheta(4\Omega'+r\Omega'')\right). \tag{4.67}$$

The derivatives of the function $f(r)$ are

$$f'(r) = \frac{2M}{r^2}\theta(r-r_0) - \frac{16\pi\rho r}{3}\theta(r_0-r), \tag{4.68}$$

$$f''(r) = -\frac{4M}{r^3}\theta(r-r_0) - \frac{16\pi\rho}{3}\theta(r_0-r) + \left(\frac{2M}{r^2}+\frac{16\pi\rho r}{3}\right)\delta(r-r_0). \tag{4.69}$$

When integrating the field equations across the shell using integrations like (4.19), only the delta functions contribute, hence only the second derivative of $f(r)$. Therefore

$$8\pi\int T^{00} = \int G^{00} = 0, \tag{4.70}$$

$$8\pi\int T^{11} = \int G^{11} = 0. \tag{4.71}$$

This means that $T^{00}=0$ at the shell as well as outside of it: the shell has no rest mass! It must therefore be seen only as a boundary of the polarized vacuum ball, not as a physical shell. Integrating the two remaining diagonal equations yields

$$8\pi\int T^{22} = \int G^{22} = \frac{M}{r_0^2} + \frac{8\pi\rho r_0}{3} = \frac{3M}{r_0^2}, \tag{4.72}$$

$$8\pi\int T^{33} = \int G^{33} = \frac{M}{r_0^2} + \frac{8\pi\rho r_0}{3} = \frac{3M}{r_0^2}, \tag{4.73}$$

which means that $T^{22} = T^{33} = 3M\delta(r - r_0)/(8\pi r^2)$ at the shell. Hence, the shell has no rest mass, but is capable of having stresses. If this shall not be completely foolish, the stresses must be seen as surface tensions belonging to the boundary between the empty space and the vacuum dominated region. We shall now integrate the remaining equation in order to obtain an expression for the induced rotation, $\Omega(r)$.

The induced rotation

The last field equation reads $G^{03} = 0$, or

$$-\frac{1}{2}\left(\sqrt{f}\sin\vartheta(4\Omega' + r\Omega'')\right) = 0. \tag{4.74}$$

Since neither $f(r)$ nor $\sin\vartheta$ equals zero everywhere, this requires

$$4\Omega' + r\Omega'' = 0. \tag{4.75}$$

The general solution of this equation is $\Omega = A/r^3 + B$, where A and B are constants of integration. The solution must be regular everywhere. Therefore $A = 0$ and $\Omega = \Omega_0$ for $r < r_0$—as before. The same argument as earlier requires $B = 0$ for $r > r_0$. Demanding continuity of Ω across the shell, we end up with

$$\Omega = \begin{cases} \Omega_0(r_0/r)^3 & \text{for } r > r_0 \\ \Omega_0 & \text{for } r < r_0 \end{cases} \tag{4.76}$$

—a result which closely resembles that of Brill and Cohen, equation (4.29).

In order to find the constant Ω_0, we must go through the same procedure as in the Brill and Cohen case. Inserting Ω and its derivatives,

$$\Omega' = -\frac{3\Omega_0 r_0^3}{r^4}\theta(r - r_0), \tag{4.77}$$

$$\Omega'' = \frac{12\Omega_0 r_0^3}{r^5}\theta(r - r_0) - \frac{3\Omega_0 r_0^3}{r^4}\delta(r - r_0), \tag{4.78}$$

into G^{03} and integrating through the shell gives

$$8\pi\int T^{03} = \int G^{03} = \frac{3}{2}\Omega_0\sin\vartheta\sqrt{f(r_0)}, \tag{4.79}$$

where $f(r_0) = 1 - 2M/r_0 = 1 - 8\pi\rho r_0^2/3$.

We shall now calculate the left hand side of this equation as we did in the Brill and Cohen case. Like before, the tensor may be written as in equation (4.37), with the same restrictions on t^{ij}. This time the components of the velocity four vector of the shell are

$$u^0 = \frac{\omega^0}{\mathrm{d}\tau} = \frac{\sqrt{f(r)}}{\sqrt{f(r) - \lambda^2}}, \tag{4.80}$$

$$u^1 = \frac{\omega^1}{\mathrm{d}\tau} = 0, \tag{4.81}$$

$$u^2 = \frac{\omega^2}{\mathrm{d}\tau} = 0, \tag{4.82}$$

$$u^3 = \frac{\omega^3}{\mathrm{d}\tau} = \frac{\lambda}{\sqrt{f(r) - \lambda^2}}, \tag{4.83}$$

where we have introduced $\lambda = r \sin\vartheta(\omega_s - \Omega)$ and $\omega_s = \mathrm{d}\varphi/\mathrm{d}t$ at the shell. Writing this as

$$u^{\hat{\mu}} = (\sqrt{f(r)}, 0, 0, \lambda)/\sqrt{f(r) - \lambda^2}, \tag{4.84}$$

it is again easy to see how the vectors ${v_{(i)}}^{\hat{\mu}}$ may be chosen:

$${v_{(1)}}^{\hat{\mu}} = (0, 1, 0, 0), \tag{4.85}$$

$${v_{(2)}}^{\hat{\mu}} = (0, 0, 1, 0), \tag{4.86}$$

$${v_{(3)}}^{\hat{\mu}} = (\lambda, 0, 0, \sqrt{f(r)})/\sqrt{f(r) - \lambda^2}. \tag{4.87}$$

Inserting these into the expression for $T^{\hat{\mu}\hat{\nu}}$ gives, correct to first order in ω_s (and hence also in λ)

$$T^{00} = \sigma, \tag{4.88}$$

$$T^{11} = t^{11}, \tag{4.89}$$

$$T^{22} = t^{22}, \tag{4.90}$$

$$T^{33} = t^{33}, \tag{4.91}$$

$$T^{03} = T^{30} = (\sigma + t^{33})\frac{\lambda}{\sqrt{f(r)}}. \tag{4.92}$$

If we compare this with the previous results, we see that $\sigma = t^{11} = 0$ and $t^{22} = t^{33} = 3M\delta(r - r_0)/(8\pi r^2)$. Inserting this into T^{03} and integrating through the shell gives

$$8\pi \int T^{03} = 8\pi \int \frac{t^{33}\lambda}{\sqrt{f(r)}} = \frac{3M(\omega_s - \Omega_0)\sin\vartheta}{r_0\sqrt{f(r_0)}}. \tag{4.93}$$

Inserting this into equation (4.79), gives

$$\Omega_0 = \frac{\omega_s}{1 + r_0 f(r_0)/2M} = \omega_s \frac{2M}{r_0}. \tag{4.94}$$

This seems to be a very nice result. Is it compatible with the result of Brill and Cohen? We will not be able to answer that question, because of the restriction we have made on the position of the shell. We demanded that the shall was placed at $r_0 = (3M/(4\pi\rho))^{1/3}$. If we try to find the limit $\rho \to 0$, *i.e.* the case considered by Brill and Cohen, the shell will disappear as well: $r_0 \to \infty$ and $\Omega_0 \to 0$. All interesting properties vanish.

We also see that *perfect dragging* appears if we put the shell at its Schwarzschild radius. However this leads to diverging stresses, as we shall se later. There are three serious problems with our solution:

1. We have a strong restriction on the radius of the shell.
2. It is not possible to take the limit $\rho \to 0$ without removing the shell.
3. The shell has no rest mass.

Later we shall find a more general solution for a rotating ball of polarized vacuum, which solves these problems, and which has the wanted properties. This will be possible using Israel's formalism for surface layers.

4.3 Israel's formalism

The attempts we made on describing a thin rotating shell in the previous section were not succesful to the wanted degree. It is possible to obtain a far better result using Israel's formalism for thin surfaces, developed in 1965 [20]. The reader is not expected to be familiar with this formalism, so before we start using it, I will summarize—without deductions—the parts of the formalism which will be of importance for us. Hence, we shall not consider the most general aspects of the theory.

Consider a space–time manifold denoted V. In this manifold there exists a space-like hypersurface Σ dividing the manifold into two parts, denoted V^+ and V^-. For these regions of the manifold we choose coordinates x^μ_+ and x^μ_- respectively. The corresponding line elements are

$$\mathrm{d}s^2_\pm = g^\pm_{\mu\nu}\mathrm{d}x^\mu_\pm \mathrm{d}x^\nu_\pm. \tag{4.95}$$

Because the hypersurface Σ is the common boundary of V^+ and V^-, it must be possible to find a parametrization for it in both systems, using the same parameters. We shall call these parameters ξ^i, where $i = 1, 2, 3$, and they will act as coordinates for the surface. From now on we are going to use Latin indices to denote components with respect to the hypersurface coordinates ξ^i, and Greek indices for the usual manifold components. The parametrization of Σ may be written

$$x^\mu_\pm = x^\mu_\pm(\xi^i). \tag{4.96}$$

The line element of the shell must be independent of whether it is seen as the boundary of V^+ or V^-. This is usually stated as "the line element belonging to the two parts of the manifold must *induce* the same intrinsic metric on Σ." When viewed as the boundary of V^+, the intrinsic metric is

$$\mathrm{d}s_\Sigma^2 = \left[g^+_{\mu\nu}\mathrm{d}x^\mu_+\mathrm{d}x^\nu_+\right]_{x^\mu_+=x^\mu_+(\xi^i)} = g^+_{\mu\nu}\frac{\partial x^\mu_+}{\partial\xi^i}\frac{\partial x^\nu_+}{\partial\xi^j}\mathrm{d}\xi^i\mathrm{d}\xi^j, \tag{4.97}$$

whereas it is

$$\mathrm{d}s_\Sigma^2 = g^-_{\mu\nu}\frac{\partial x^\mu_-}{\partial\xi^i}\frac{\partial x^\nu_-}{\partial\xi^j}\mathrm{d}\xi^i\mathrm{d}\xi^j \tag{4.98}$$

when viewed as the boundary of V^-. The two expressions must be equal, whence

$$\mathrm{d}s_\Sigma^2 = g_{ij}\mathrm{d}\xi^i\mathrm{d}\xi^j, \tag{4.99}$$

where

$$g_{ij} \equiv g^+_{\mu\nu}\frac{\partial x^\mu_+}{\partial\xi^i}\frac{\partial x^\nu_+}{\partial\xi^j} = g^-_{\mu\nu}\frac{\partial x^\mu_-}{\partial\xi^i}\frac{\partial x^\nu_-}{\partial\xi^j} \tag{4.100}$$

are the components of the induced metric tensor on Σ with respect to the surface coordinates ξ^i.

Energy–momentum tensor on Σ

Let n^μ denote the unit normal of Σ, directed from V^- to V^+. The covariant derivative of this vector using the intrinsic coordinates, ξ^i, describes the way Σ curves in V. It is therefore natural to introduce the *extrinsic curvature tensor*

$$K_{ij} = -n_{i;j} = -n_{i,j} + n_k\Gamma^k_{ij}, \tag{4.101}$$

where Γ^k_{ij} are the Christoffel symbols assosiated with the intrinsic metric of Σ. The curvature tensor is not necessarily the same on the two sides of the surface. Let K^+_{ij} denote the value of K_{ij} in V^+, and K^-_{ij} its value in V^-. Introduce $K^\pm = g^{ij}K^\pm_{ij}$ and

$$[K_{ij}] = K^+_{ij} - K^-_{ij}, \tag{4.102}$$

$$[K] = K^+ - K^-. \tag{4.103}$$

According to Israel, the energy–momentum tensor of the shell is

$$S_{ij} = -\frac{1}{8\pi}([K_{ij}] - g_{ij}[K]), \tag{4.104}$$

or

$$S^i{}_j = -\frac{1}{8\pi}([K^i{}_j] - \delta^i_j[K]). \tag{4.105}$$

The components $S^i{}_j$ may be arranged as a matrix, whoose eigenvalues $\lambda_{(k)}$ (the index in parenthesis is only a tag and does *not* denote the component of any vector) may be found from solving

$$|S^i{}_j - \delta^i_j \lambda_{(k)}| = 0, \tag{4.106}$$

where $|A|$ denotes the determinant of A. The three corresponding orthogonal eigenvectors, $v_{(k)}{}^i$—one time-like, two space-like are given by

$$S^i{}_j v_{(k)}{}^j = \lambda_{(k)} v_{(k)}{}^i. \tag{4.107}$$

From now on we demand that the eigenvectors are normalized. Since they are orthogonal, they form a basis triad for the hypersurface. The time-like eigenvector is in fact the velocity four vector of the hypersurface—denoted u^i—whence the eigenvectors form an orthonormal comoving basis.

The eigenvalue corresponding to the time-like eigenvector is the proper energy density of the surface, σ. The two remaining eigenvalues represent the negative of the stresses in directions given by the corresponding eigenvectors:

$$\lambda_{u^i} = \sigma, \tag{4.108}$$

$$\lambda_{v_{(k)}{}^i} = -p_{(k)}, \quad \text{where } k = 1, 2. \tag{4.109}$$

It has been showed by Lichnerowicz that the energy–momentum tensor of the surface in this notation may be written as

$$S^{ij} = \sigma u^i u^j + \sum_{k=1}^{2} p_{(k)} v_{(k)}{}^i v_{(k)}{}^j. \tag{4.110}$$

4.4 The rotating shell revisited

We are now ready to use Israel's formalism to describe the rotating spherical shell containing polarized vacuum. In this situation we let Σ denote the shell, V^+ the region of the space–time outside the shell, V^- its interior, and ω_s the *coordinate* angular velocity of the shell. The induced rotation will be found using the following 5-step procedure:

1. Find suitable coordinates and calculate the extrinsic and intrinsic metric tensors.

2. Use the field equations to find out as much as possible about the induced rotation, Ω.

3. Find the energy–momentum tensor for the shell, S^{ij}, as given by Israel's formalism.

4. Use the formula (4.110) to find an expression for S^{ij} containing ω_s.
5. The results of steps 3 and 4 must be equal. Comparing them, we will find the rest mass density of the shell σ, the stresses $p_{(k)}$, and the relation between the rotation of the shell ω_s, and the induced rotation Ω.

As before, the calculations will be carried out to first order in ω_s.

4.4.1 Coordinates and metrics

The situation is exactly the same as earlier. Consider first the non-rotating case. In V^+ there is Schwarzschild space–time, and if we use t, r, ϑ and φ as coordinates, the line element is

$$\mathrm{d}s_+^2 = (1-2M/r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1-2M/r} - r^2(\mathrm{d}\vartheta^2 + \sin^2\!\vartheta\,\mathrm{d}\varphi^2), \tag{4.111}$$

where M is the gravitational mass of the spherically symmetric system. In these coordinates the radial position of the shell is $r = r_0$, and the region V^+ corresponds to $r > r_0$. The other 'half' of the space–time—the region V^-—is described by the de Sitter metric. In this region we use coordinates T, R, Θ and Φ, which so far are unrelated to the coordinates of V^+. The metric of V^- is

$$\mathrm{d}s_-^2 = (1-8\pi\rho R^2/3)\mathrm{d}T^2 - \frac{\mathrm{d}R^2}{1-8\pi\rho R^2/3} - R^2(\mathrm{d}\Theta^2 + \sin^2\Theta\,\mathrm{d}\Phi^2), \tag{4.112}$$

and is valid for $R < R_0$, where R_0 is the position of the shell as described from V^-.

There are plenty of possible choices for coordinates on the shell. Two of them are more natural than the others: We may choose t, ϑ and φ, or T, Θ and Φ. This gives two different versions of the *intrinsic* metric on Σ,

$$\mathrm{d}s_{\Sigma-}^2 = (1-8\pi\rho R_0^2)\mathrm{d}T^2 - R_0^2(\mathrm{d}\Theta^2 + \sin^2\Theta\,\mathrm{d}\Phi^2), \tag{4.113}$$

$$\mathrm{d}s_{\Sigma+}^2 = (1-2M/r_0)\mathrm{d}t^2 - r_0^2(\mathrm{d}\vartheta^2 + \sin^2\!\vartheta\,\mathrm{d}\varphi^2). \tag{4.114}$$

These are the line elements of the *same* surface, hence they must be equal. This indicates that we can *choose* the following relation between the outer and inner coordinates:

$$R_0 = r_0, \tag{4.115}$$

$$\Theta = \vartheta, \tag{4.116}$$

$$\Phi = \varphi, \tag{4.117}$$

$$T = \sqrt{\frac{1-2M/r_0}{1-8\pi\rho r_0^2/3}}\; t \equiv \sqrt{a}\, t. \tag{4.118}$$

Since $R = r = r_0$ at the shell, we can take them to be equal everywhere without loss of generality. We have now found a set of coordinates which covers both V^+ and V^-.

Like before, the possibility of a rotating shell may be inserted into the line element as an additional function, $\Omega(r)$. We put the same restrictions on Ω as before. Hence the line element of our space–time with a slowly rotating shell is

$$\mathrm{d}s^2 = g(r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{f(r)} - r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta(\mathrm{d}\varphi - \Omega(r)\mathrm{d}t)^2\right), \tag{4.119}$$

$$f(r) = \begin{cases} 1 - 8\pi\rho r^2/3 & \text{for } r < r_0 \\ 1 - 2M/r & \text{for } r > r_0 \end{cases}, \tag{4.120}$$

$$g(r) = \begin{cases} a(1 - 8\pi\rho r^2/3) & \text{for } r < r_0 \\ 1 - 2M/r & \text{for } r > r_0 \end{cases}, \tag{4.121}$$

$$a = \frac{1 - 2M/r_0}{1 - 8\pi\rho r_0^2/3}, \tag{4.122}$$

which gives the induced metric at Σ:

$$\mathrm{d}s_\Sigma^2 = (1 - 2M/r_0)\mathrm{d}t^2 - r_0^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta(\mathrm{d}\varphi - \Omega(r_0)\mathrm{d}t)^2\right) \tag{4.123}$$

when t, ϑ and φ are used as coordinates on the shell.

For convenience, we introduce the short hand notation

$$\beta_D = \sqrt{1 - \frac{8\pi\rho r^2}{3}}, \tag{4.124}$$

$$\beta_S = \sqrt{1 - \frac{2M}{r}}. \tag{4.125}$$

Here D suggests 'de Sitter,' and S 'Schwarzschild.' We shall also use β_{D0} and β_{S0} for the values of β_D and β_S at $r = r_0$, respectively.

4.4.2 The induced rotation

To find out something about the induced rotation, we start with setting up the field equations corresponding to the line element (4.119). A convenient local orthonormal frame is

$$\boldsymbol{\omega}^0 = \sqrt{g(r)}\,\boldsymbol{d}t, \tag{4.126}$$

$$\boldsymbol{\omega}^1 = \frac{\boldsymbol{d}r}{\sqrt{f(r)}}, \tag{4.127}$$

$$\boldsymbol{\omega}^2 = r\,\boldsymbol{d}\vartheta, \tag{4.128}$$

$$\boldsymbol{\omega}^3 = r\,\sin\vartheta(\boldsymbol{d}\varphi - \Omega(r)\boldsymbol{d}t). \tag{4.129}$$

In this frame, CARTAN tells us that the components of the Einstein tensor to first order in ω_s, Ω and its derivatives are

$$G^{00} = \frac{1 - f - rf'}{r^2}, \tag{4.130}$$

$$G^{11} = \frac{f-1}{r^2} + \frac{fg'}{rg}, \tag{4.131}$$

$$G^{22} = G^{33} = \frac{2g^2 f' + 2fgg' + rgf'g' - rfg'^2 - 2rfgg''}{4rg^2}, \tag{4.132}$$

$$G^{03} = G^{30} = \frac{\sin\vartheta}{4g^{3/2}}(-8fg\Omega' - rgf'\Omega' + rfg'\Omega' - 2rfg\Omega''). \tag{4.133}$$

Like before, the diagonal components of the Einstein tensor are unaffected by the rotation. The non-diagonal component gives us an equation for Ω: $G^{03} = 0$. We notice that $f(r) = Ag(r)$, where $A = 1$ for $r > r_0$ and $A = a$ for $r < r_0$. The equation therefore reduces to

$$4\Omega' + r\Omega'' = 0 \tag{4.134}$$

in both V^+ and V^-. We have seen this equation before (4.75), and still the solution is

$$\Omega = \begin{cases} \Omega_P (r_0/r)^3 & \text{for } r > r_0 \\ \Omega_P & \text{for } r < r_0 \end{cases}, \tag{4.135}$$

when we demand nice behavior at $r = 0$ and $r = \infty$ and continuity across the shell. Ω_P is a constant of integration, and P suggests 'polarized vacuum,' in order to distinguish it from the case considered by Brill & Cohen. The result from the last section was more general than we could expect. However, the value of Ω_P will depend on the radial position of the shell, which was fixed in the previous section.

4.4.3 The shell described using Israel's formalism

Knowing the metric tensor of the space–time manifold and the parametrization of the singular surface, it is straight forward to compute its energy–momentum tensor using the formalism of Israel. Because it is an illustrating example of how to use the formalism, I will present a detailed calculation here.

Before we can find S^{ij}, we must know $K^i{}_j$ on both sides of the shell. To avoid confusion, we shall calculate them separately.

Extrinsic curvature in V^+

Knowledge of both covariant and contravariant components of the metric tensor in this region will turn out to be handy. To first order in Ω they are

$$\begin{aligned} g_{tt} &= \beta_S^2 \\ g_{rr} &= -\beta_S^{-2} \\ g_{\vartheta\vartheta} &= -r^2 \\ g_{\varphi\varphi} &= -r^2\sin^2\vartheta \\ g_{\varphi t} &= \Omega\, r^2\sin^2\vartheta \end{aligned} \qquad \text{and} \qquad \begin{aligned} g^{tt} &= \beta_S^{-2} \\ g^{rr} &= -\beta_S^2 \\ g^{\vartheta\vartheta} &= -r^{-2} \\ g^{\varphi\varphi} &= -(r^2\sin\vartheta)^{-2} \\ g^{\varphi t} &= \Omega\,\beta_S^{-2} \end{aligned} \tag{4.136}$$

respectively. The contravariant ones are found by matrix inversion. The normal vector of the surface is

$$n^\mu = (0, \beta_S, 0, 0), \tag{4.137}$$

or, covariantly,

$$n_\mu = g_{\mu\nu}n^\nu = (0, -\beta_S^{-1}, 0, 0). \tag{4.138}$$

We will also need some of the Christoffel symbols belonging to the given line element. These are (as given by CARTAN)

$$\Gamma^r_{\vartheta\vartheta} = -r\beta_S^2 \tag{4.139}$$

$$\Gamma^r_{\varphi\varphi} = -r\beta_S^2\sin^2\vartheta, \tag{4.140}$$

$$\Gamma^r_{tt} = \beta_S^2\frac{M}{r^2}, \tag{4.141}$$

$$\Gamma^r_{\varphi t} = r\beta_S^2\sin^2\vartheta\left(\Omega + \frac{1}{2}r\Omega'\right). \tag{4.142}$$

Calculating the extrinsic curvature, we find that $n_{i,j} = 0$, so

$$K^+_{ij} = -n_\mu\Gamma^\mu_{ij} = -\beta_S^{-1}\Gamma^r_{ij}, \tag{4.143}$$

whence

$$K^+_{\vartheta\vartheta} = r\beta_S, \tag{4.144}$$

$$K^+_{\varphi\varphi} = r\beta_S\sin^2\vartheta, \tag{4.145}$$

$$K^+_{tt} = -\beta_S\frac{M}{r^2}, \tag{4.146}$$

$$K^+_{\varphi t} = -r\beta_S\sin^2\vartheta\left(\Omega + \frac{1}{2}r\Omega'\right), \tag{4.147}$$

which gives the mixed components

$$K^{\varphi}_{+\varphi} = K^{\vartheta}_{+\vartheta} = -\frac{\beta_S}{r}, \tag{4.148}$$

$$K^t_{+t} = -\beta_S^{-1}\frac{M}{r^2}, \tag{4.149}$$

$$K^{\varphi}_{+t} = \frac{\beta_S}{r}\left(\Omega + \frac{1}{2}r\Omega'\right) - \frac{\Omega M}{r^2\beta_S}, \tag{4.150}$$

when calculating to first order in Ω and its derivatives. What we need is of course the value of K at the shell, *i.e.* at $r = r_0$. This gives the above result with $r \mapsto r_0$, $\Omega \mapsto \Omega_P$ and $\Omega' \mapsto -3\Omega_P/r_0$ (the value of Ω' at the outer side of the shell). For the ${}^{\varphi}{}_{t}$-component this gives

$$K^{\varphi}_{+t} = -\Omega_P \left(\frac{M}{r_0^2 \beta_{S0}} + \frac{\beta_{S0}}{2r_0} \right). \tag{4.151}$$

Extrinsic curvature in V^-

The curvature tensor in V^- can be calculated in exactly the same way as in V^+. The components of the metric tensor to first order in Ω are

$$\begin{aligned} g_{tt} &= a\beta_D^2 \\ g_{rr} &= -\beta_D^{-2} \\ g_{\vartheta\vartheta} &= -r^2 \\ g_{\varphi\varphi} &= -r^2 \sin^2\vartheta \\ g_{\varphi t} &= r^2 \sin^2\vartheta \end{aligned} \quad \text{and} \quad \begin{aligned} g^{tt} &= (a\beta_D^2)^{-1} \\ g^{rr} &= -\beta_D^2 \\ g^{\vartheta\vartheta} &= -r^{-2} \\ g^{\varphi\varphi} &= -(r\sin\vartheta)^{-2} \\ g^{\varphi t} &= \Omega(a\beta_D^2)^{-1} \end{aligned} \tag{4.152}$$

which together with the unit normal vector

$$n^\mu = (0, \beta_D, 0, 0) \quad \Rightarrow \quad n_\mu = (0, -\beta_D^{-1}, 0, 0) \tag{4.153}$$

and the Christoffel symbols

$$\Gamma^r_{\vartheta\vartheta} = -r\beta_D^2, \tag{4.154}$$
$$\Gamma^r_{\varphi\varphi} = -r\beta_D^2 \sin^2\vartheta, \tag{4.155}$$
$$\Gamma^r_{tt} = -ra\beta_D^2 \frac{8\pi\rho}{3}, \tag{4.156}$$
$$\Gamma^r_{\varphi t} = r\Omega\beta_D^2 \sin^2\vartheta, \tag{4.157}$$

gives the wanted curvature tensor

$$K^-_{\vartheta\vartheta} = r\beta_D, \tag{4.158}$$
$$K^-_{\varphi\varphi} = r\beta_D \sin^2\vartheta, \tag{4.159}$$
$$K^-_{tt} = ra\beta_D \frac{8\pi\rho}{3}, \tag{4.160}$$
$$K^-_{\varphi t} = -r\Omega\beta_D \sin^2\vartheta \tag{4.161}$$

or, if we raise one index and insert the values at $r = r_0$ (where $\Omega' = 0$),

$$K^{\varphi}_{-\varphi} = K^{\vartheta}_{-\vartheta} = -\frac{\beta_{D0}}{r_0}, \tag{4.162}$$
$$K^{t}_{-t} = \frac{8\pi\rho r_0}{3\beta_{D0}}, \tag{4.163}$$
$$K^{\varphi}_{-t} = \Omega_P \left[\frac{\beta_{D0}}{r_0} + \frac{8\pi\rho r_0}{3\beta_{D0}} \right]. \tag{4.164}$$

The energy–momentum tensor

It is now a simple task to calculate the energy–momentum tensor from equation (4.105). This results in

$$S^t{}_t = \frac{1}{4\pi}[K^\vartheta{}_\vartheta] = \frac{\beta_{D0} - \beta_{S0}}{4\pi r_0}, \tag{4.165}$$

$$\begin{aligned} S^\varphi{}_\varphi = S^\vartheta{}_\vartheta &= \frac{1}{8\pi}\left([K^t{}_t] + [K^\vartheta{}_\vartheta]\right) \\ &= \frac{1}{2}S^t{}_t - \frac{1}{8\pi}\left(\frac{M}{r_0^2\beta_{S0}} + \frac{8\pi\rho r_0}{3\beta_{D0}}\right), \end{aligned} \tag{4.166}$$

$$S^\varphi{}_t = -\frac{1}{8\pi}[K^\varphi{}_t] = \frac{\Omega_P}{8\pi}\left(\frac{\beta_{S0}}{2r_0} + \frac{\beta_{D0}}{r_0} - [K^t{}_t]\right). \tag{4.167}$$

Raising the indices of the diagonal components gives

$$S^{tt} = \frac{\beta_{D0} - \beta_{S0}}{4\pi r_0 \beta_{S0}^2}, \tag{4.168}$$

$$S^{\vartheta\vartheta} = \frac{\beta_{S0} - \beta_{D0} + r_0\left(\frac{M}{r_0^2\beta_{S0}} + \frac{8\pi\rho r_0}{3\beta_{D0}}\right)}{8\pi r_0^3}, \tag{4.169}$$

$$S^{\varphi\varphi} = \frac{S^{\vartheta\vartheta}}{\sin^2\vartheta}. \tag{4.170}$$

We leave the ${}^\varphi{}_t$-component while carrying out step four of our procedure, which shall lead to an alternative version of S^{ij}.

4.4.4 Lichnerowicz' expression for the energy–momentum tensor

In order to relate the components of the energy–momentum tensor to the physical properties of the shell, we may use the expression of Lichnerowicz, equation (4.110). We need to find the vectors u^i and ${v_{(k)}}^i$. Here u^i is the velocity vector of the shell:

$$u^i = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}\vartheta}{\mathrm{d}\tau}, \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)_{\text{along motion}}. \tag{4.171}$$

At the shell $\mathrm{d}\vartheta/\mathrm{d}\tau = 0$ and $\mathrm{d}\tau = \sqrt{\beta_{S0}^2)\mathrm{d}t^2 - r_0^2\sin^2\vartheta(\mathrm{d}\varphi - \Omega_P\mathrm{d}t)^2}$, which gives

$$u^i = (1, 0, \omega_s)/\sqrt{\beta_{S0}^2 - r_0^2\sin^2\vartheta(\omega_s - \Omega_P)^2}. \tag{4.172}$$

The two vectors ${v_{(k)}}^i$ must be chosen so that they are mutually orthogonal, normal, and orthogonal to u^i. It is easy to convince oneself that one of these vectors may be chosen as

$${v_{(1)}}^i = (0, 1/r_0, 0). \tag{4.173}$$

The components of ${v_{(2)}}^i$ are found by solving the simultaneous equations

$$g_{ij}{v_{(2)}}^i{v_{(2)}}^j = -1, \tag{4.174}$$

$$g_{ij}{v_{(2)}}^i{v_{(1)}}^j = 0, \tag{4.175}$$

$$g_{ij}{v_{(2)}}^i u^j = 0. \tag{4.176}$$

The results are, to first order in Ω:

$$u^i = (1, 0, \omega_s)/\beta_{S0}, \tag{4.177}$$

$${v_{(1)}}^i = (0, 1/r_0, 0), \tag{4.178}$$

$${v_{(2)}}^i = (r_0{\beta_{S0}}^{-2}(\omega_s - \Omega_P)\sin\vartheta, 0, (r_0\sin\vartheta)^{-1}), \tag{4.179}$$

which leads to the following expressions for the energy–momentum tensor components

$$S^{tt} = \sigma{\beta_{S0}}^{-2}, \tag{4.180}$$

$$S^{\vartheta\vartheta} = \frac{p_{(1)}}{r_0^2}, \tag{4.181}$$

$$S^{\varphi\varphi} = \frac{p_{(2)}}{r_0^2\sin^2\vartheta}, \tag{4.182}$$

$$S^{\varphi t} = {\beta_{S0}}^{-2}\left((\sigma + p_{(2)})\omega_s - p_{(2)}\Omega_P\right). \tag{4.183}$$

4.4.5 The properties of the rotating shell

Comparing Israel's and Lichnerowicz' values for the energy–momentum tensor of the shell, we can find its properties. Looking at S^{tt}, we see that the rest mass density σ of the shell is given as

$$\sigma = \frac{\beta_{D0} - \beta_{S0}}{4\pi r_0} = {S^t}_t, \tag{4.184}$$

while $S^{\vartheta\vartheta}$ and $S^{\varphi\varphi}$ give the stresses:

$$\begin{aligned} p_{(1)} = p_{(2)} &= -\frac{\sigma}{2} + \frac{1}{8\pi}\left(\frac{M}{r_0^2\beta_{S0}} + \frac{8\pi\rho r_0}{3\beta_{D0}}\right) \\ &= -{S^\vartheta}_\vartheta = -{S^\varphi}_\varphi. \end{aligned} \tag{4.185}$$

Using this information, it is easy to raise the lower index of Israel's ${S^\varphi}_t$, in order to obtain

$$S^{\varphi t} = \Omega_P{\beta_{S0}}^{-2}\left(\frac{\sigma}{2} + \frac{1}{8\pi}\left(\frac{\beta_{S0}}{2r_0} + \frac{\beta_{D0}}{r_0}\right)\right). \tag{4.186}$$

Comparing this with the result we obtained using the expression of Lichnerowicz, we have an equation relating ω_s and Ω_P, whoose solution may be written

$$\frac{\Omega_P}{\omega_s} = \frac{8\pi(\sigma + p)}{4\pi(\sigma + 2p) - (\beta_{S0}/(2r_0) + \beta_{D0}/r_0)}, \tag{4.187}$$

or, equivalently,

$$\frac{\Omega_P}{\omega_s} = \frac{\beta_{S0} - \beta_{D0}\left(1 - \frac{3M}{r_0}\right)}{\beta_{S0} + \beta_{D0}/2} \tag{4.188}$$

—a result that has to be checked out against certain special cases.

4.5 Special cases

If the result (4.188) is correct, it has to contain the rotating shell of Brill & Cohen and the one considered in section 2 of this chapter as special cases. Let us check this out to assure ourselves that the result is correct.

Putting the shell at $r_0 = (3M/8\pi\rho)^{1/3}$

This special radial position of the shell assures that $\beta_{S0} = \beta_{D0}$ and that the components of the metric tensor are continous across the shell. Inserting this into the previous results, we obtain

$$\frac{\Omega_P}{\omega_s} = \frac{2M}{r_0}, \tag{4.189}$$

$$\sigma = 0. \tag{4.190}$$

$$\tag{4.191}$$

We understand that this is a very extreme situation, since the rest mass density of the shell vanishes. The induced rotation is then a consequence of the stresses belonging to the shell, and possibly the energy contained by the vacuum in its interior. The result is exactly the same as the one in section 2 of this chapter, equation (4.94), which indicates that our result is correct.

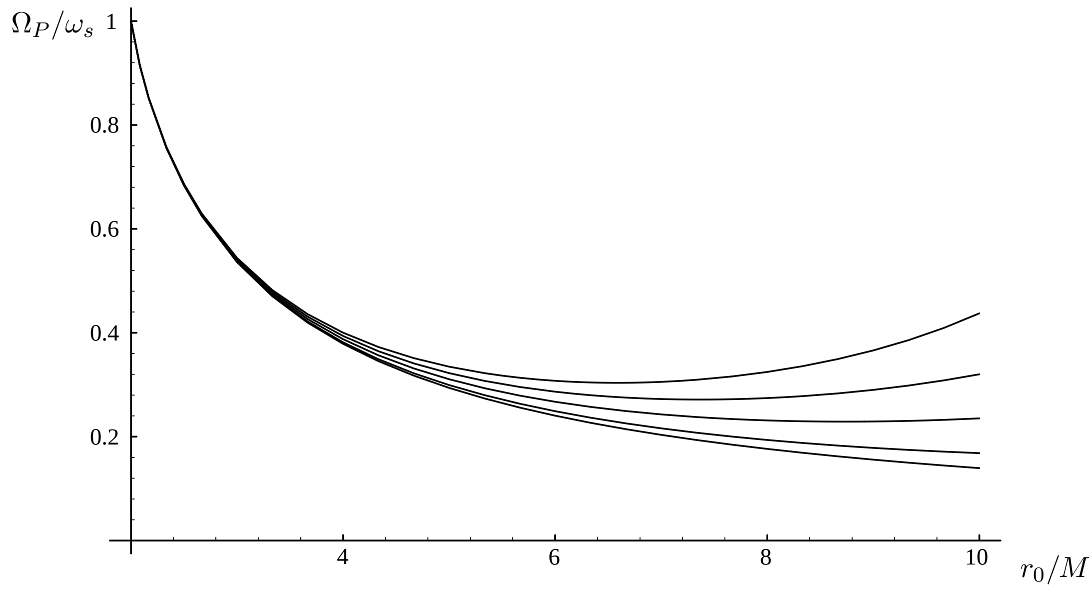
Comparing with the result of Brill & Cohen

Brill & Cohen considered a rotating shell with vanishing vacuum energy density in its interior. Inserting $\rho = 0$ and $\beta_{D0} = 1$ into our results gives

$$\frac{\Omega_P}{\omega_s} = \frac{\beta_{S0} - \left(1 - \frac{3M}{r_0}\right)}{\beta_{S0} + \frac{1}{2}}, \tag{4.192}$$

$$\sigma = \frac{1 - \beta_{S0}}{4\pi r_0}. \tag{4.193}$$

At first glance this doesn't look much like Brill & Cohen's result. However, we must remember that they used *isotropic* coordinates for the Schwarzschild



Figur 4.1: Angular dragging velocity in the interior of the shell for different values of the energy density

geometry, while we have been using the standard ones. Inserting the relation between them, equation (4.2), into our result gives (after some algebra)

$$\frac{\Omega_P}{\omega_s} = \left(1 + \frac{3}{4M}\frac{(\tilde{r}_0 - M/2)^2}{\tilde{r}_0 - M/4}\right)^{-1}, \tag{4.194}$$

which is identical with the result of Brill & Cohen, equation (4.56).

We have thus shown that our result is consistent with the two known special cases in the interior of the shell. Brill & Cohen's result is also consistent with the result of Thirring, whence our result must be so, as well. But, does our result equal the result of Brill & Cohen outside the shell also? Their result in this region is

$$\Omega = \Omega_B \left(\frac{\tilde{r}_0\psi_0^2}{\tilde{r}\psi^2}\right)^3 = \Omega_B \left(\frac{\tilde{r}_0(1 - M/2\tilde{r}_0)^2}{\tilde{r}(1 - M/2\tilde{r})^2}\right)^3 = \Omega_B \left(\frac{r_0}{r}\right)^3, \tag{4.195}$$

where the relation between r and $\tilde{r}$ has been inserted. This shows that our result with $\rho = 0$ is *identical* to the result of Brill & Cohen.

In figure 4.1 I have plotted the relative angular dragging velocity for different energy densities. The lower curve corresponds to $\rho = 0$. We see that the dragging velocity is larger for polarized vacuum than for empty space, especially for the larger r_0.

4.6 Perfect dragging conditions and consequences

An interesting question related to the solution is whether *perfect dragging* is possible, *i.e.* whether the induced rotation in the interior of the shell can equal the angular velocity of the shell. Brill & Cohen found that in their case perfect dragging occured for $r_0 = 2M$, but we will show here that this is an unphysical situation.

First, we observe that the induced velocity may be written

$$\frac{\Omega_P}{\omega_s} = 1 - \frac{3\beta_{D0}\beta_{S0}^2}{2\beta_{S0} + \beta_{D0}} \equiv 1 - B. \tag{4.196}$$

For $M \geq 0$ and $\rho \geq 0$ we must have $B \geq 0$ and $\Omega_P/\omega_s \leq 1$, *i.e.* we cannot have *over*-perfect dragging.

Perfect dragging occurs only when $B = 0$, which in turn can be the case only if $r_0 = 2M$ or $r_0 = \sqrt{3/(8\pi\rho)}$. Let us calculate the properties of the shell corresponding to these two special cases.

$$r_0 = 2M \quad \Rightarrow \quad \sigma = \frac{\beta_{D0}}{4\pi r_0}, \tag{4.197}$$

$$p \to +\infty, \tag{4.198}$$

$$r_0 = \sqrt{\frac{3}{8\pi\rho}} \quad \Rightarrow \quad \sigma = -\frac{\beta_{S0}}{4\pi r_0}, \tag{4.199}$$

$$p \to +\infty. \tag{4.200}$$

In both cases the stresses of the shell diverge, and the situations must be considered unphysical.

We also see that the last case leads to a *negative* energy density σ, as measured by an observer at rest in the coordinate system. This is as unphysical as anything, and we must find out when this can happen. The expression for σ, equation (4.184), shows that we must have $\beta_{D0} > \beta_{S0}$, or

$$r_0 \leq \left(\frac{3M}{4\pi\rho}\right)^{1/3} \tag{4.201}$$

in order to assure that $\sigma \geq 0$. The limiting case (with equality in (4.201)) is exactly the one considered in section 2 of this chapter. This is yet another reason that this case was a very special one.

From this discussion it is clear that perfect dragging by a stationary shell cannot occur in a situation which resembles anything in the "real world."

4.7 Does the vacuum rotate?

We have found that the existence of polarized vacuum in the interior of the shell affects the magnitude of the induced rotation. It is then natural to ask what the reason for this change is. It is not only the energy density ρ which is different in the two cases. In both cases the shell was constructed so that the situation should be static. In other words we had to change its rest mass density and the stresses to be able to 'insert' the vacuum energy. What is then the reason for the change in induced rotation inside the shell: the polarized vacuum or the shell?

It is reasonable to guess that both the shell and the vacuum give a contribution to the change in the induced rotation in the interior of the shell, *i.e.* if we introduce

$$\Delta\Omega_0 \equiv \Omega_P - \Omega_B, \tag{4.202}$$

we assume that it may be written as

$$\Delta\Omega_0 = (\Delta\Omega_0)_{\text{shell}} + (\Delta\Omega_0)_{\text{vacuum}}. \tag{4.203}$$

What really interests us is the contribution (if any) given by the polarized vacuum. If we shall be able to exhibit the effects of the vacuum, we must look at the situation where the shell has the same properties as in the Brill & Cohen case, while there is polarized vacuum in its interior. In this situation the shell will not be stationary because of the repulsive gravity of the polarized vacuum in its interior. However, at one instant of time the radial velocity of the shell may vanish. Let this instant be at a point of time t_0, and the radial position of the shell $r_0(t_0)$.

Since $r_0 = r_0(t)$, all constants of the previous discussion must be time dependent. We may try with a line element of the form

$$\mathrm{d}s^2 = g(r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{f(r)} - r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta(\mathrm{d}\varphi - \Omega(r,t)\mathrm{d}t)^2\right), \tag{4.204}$$

$$f(r) = \begin{cases} 1 - 8\pi\rho r^2/3 & \text{for } r < r_0(t) \\ 1 - 2M/r & \text{for } r > r_0 \end{cases}, \tag{4.205}$$

$$g(r) = \begin{cases} a(t)(1 - 8\pi\rho r^2/3) & \text{for } r < r_0(t) \\ 1 - 2M/r & \text{for } r > r_0(t) \end{cases}, \tag{4.206}$$

$$a(t) = \frac{1 - 2M/r_0(t)}{1 - 8\pi\rho r_0(t)^2/3}. \tag{4.207}$$

This ansatz gives the same Einstein tensor as before, except that

$$G^1{}_3 = \frac{r\sin\vartheta(\dot{a}\Omega' - 2a\dot{\Omega}')}{4\beta_D a^2}, \tag{4.208}$$

$$G^3{}_4 = -\frac{\beta_D \sin\vartheta(4\Omega' + r\Omega'')}{2\sqrt{a}}. \tag{4.209}$$

Here the dot denotes differentiation with respect to time, and the prime denotes differentiation with respect to r.

Suppose now that the induced rotation, Ω may be factorized

$$\Omega(r,t) = \tilde{\Omega}(r)h(t). \tag{4.210}$$

The equations corresponding to $G^1{}_3 = G^3{}_4 = 0$ then reads

$$4\tilde{\Omega}' + r\tilde{\Omega}'' = 0, \tag{4.211}$$
$$\dot{a}h - 2a\dot{h} = 0. \tag{4.212}$$

The first one is solved by $\tilde{\Omega} = A/r^3 + B$, while the second gives $h = Ca^2$, where A, B and C are constants of integration. Hence the induced rotation is

$$\Omega = Ca(t)^2 \left(\frac{A}{r^3} + B\right). \tag{4.213}$$

In order to obtain nice behaviour at $r \to \infty$ and $r \to 0$, and continuity across the shell, we must have $\Omega = \Omega_0(t)$ inside the shell, and $B = 0$ outside. Hence

$$\Omega = \begin{cases} \Omega_0(r_0/r)^3 & \text{for } r > r_0 \\ \Omega_0 & \text{for } r < r_0 \end{cases}, \tag{4.214}$$

where Ω_0 and r_0 are functions of t. The value of Ω_0 at t_0 (when the shell is at rest), is denoted $\Omega_{P/B}$. The 'P/B' suggests a rotating shell containing $p$olarized vacuum, and that the shell has Brill & Cohen properies. Using Israel's formalism to calculate the extrinsic curvature of the shell, we arrive at the same expressions as earlier, except that time dependent functions have taken the places of r_0 and Ω_B.

Because the radial velocity of the shell vanishes at t_0, its energy–momentum tensor is, like before,

$$S^{tt} = \sigma \beta_{S0}{}^{-2}, \tag{4.215}$$
$$S^{\vartheta\vartheta} = \frac{p}{r_0^2}, \tag{4.216}$$
$$S^{\varphi\varphi} = \frac{p}{r_0^2 \sin^2\vartheta}, \tag{4.217}$$
$$S^{\varphi t} = \beta_{S0}{}^{-2}\left((\sigma + p)\omega_s - p\Omega_P\right), \tag{4.218}$$

where σ and p have the same values as the ones describing the shell of Brill & Cohen:

$$\sigma = \frac{1 - \beta_{S0}}{4\pi r_0}, \tag{4.219}$$
$$p = -\frac{\sigma}{2} + \frac{M}{8\pi r_0^2 \beta_{S0}}. \tag{4.220}$$

Now the ${}^{\varphi t}$-component of the energy–momentum tensor for the shell as obtained by Israel's formalism may be written

$$S^{\varphi t} = \frac{\Omega_{P/B}}{8\pi r_0 \beta_{S0}^2}(2\beta_{D0} - \beta_{S0}/2). \tag{4.221}$$

Inserting this into the above expression gives

$$\frac{\Omega_{P/B}}{\omega_s} = \frac{\beta_{S0} - (1 - 3M/r_0)}{(2\beta_{D0} - 1)\beta_{S0} + 1/2}. \tag{4.222}$$

This is the dragging velocity of a rotating Brill & Cohen-shell containing polarized vacuum. If we let the energy density ρ vanish, we should arrive at the result of Brill & Cohen again. We see that this actually is the case, since this means that $\beta_{D0} = 1$, whence $2\beta_{D0} - 1 = 1$. Notice that perfect dragging occurs for $r_0 = 2M$, but again the stresses p, diverge.

Actually, $2\beta_{D0} - 1 \leq 1$ always, which shows that the dragging velocity of the shell containing polarized vacuum always is greater than that of the empty shell. This is a strong indication that both the shell and the vacuum contribute to the inertial dragging. But, if the vacuum itself gives rise to inertial dragging effects, it must be moving!

Critical examination of $\Omega_{P/B}$.

There is however one problem associated with the induced dragging velocity (4.222). According to this solution it is possible to obtain *more* than perfect dragging, *i.e.* it is possible to have $\Omega_{P/B}/\omega_s > 1$. We don't want this situation to be achievable, so let us examine where this happens.

There are some restrictions on the radial position of the shell, namely that the βs have to be real. This means that we have to restrict ourselves to

$$2M \leq r_0 \leq \sqrt{\frac{3}{8\pi\rho}}, \tag{4.223}$$

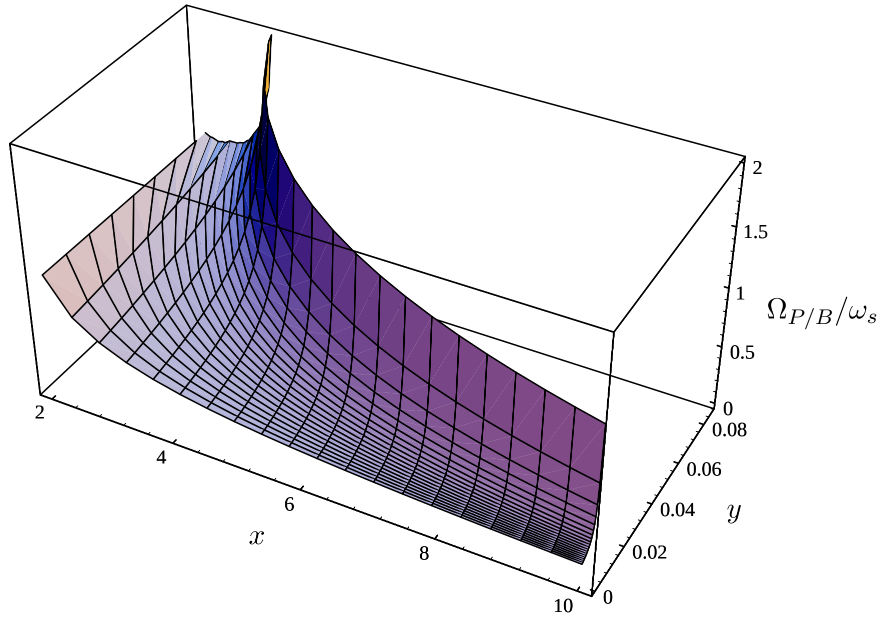
which again gives a restriction on ρ and M: $\sqrt{3/(8\pi\rho)} \geq 2M$.

Introduce now the scaled parametres $x = r_0/M$ and $y = 8\pi\rho M^2/3$. The induced angular velocity in the interior of the shell may then be written

$$\frac{\Omega_{P/B}}{\omega_s} = \frac{\sqrt{1 - 2/x} - (1 - 3/x)}{\left(2\sqrt{1 - yx^2} - 1\right)\sqrt{1 - 2/x} + 1/2}, \tag{4.224}$$

and the above restriction on the radius of the shell is

$$2 \leq x \leq y^{-1/2}. \tag{4.225}$$



Figur 4.2: Angular dragging velocity for a Brill & Cohen-shell containing polarized vacuum

It is now easy to see that perfect dragging occurs for

$$y = \frac{3(6 - 3x + 8x\sqrt{1 - 2/x})}{16x^3}, \tag{4.226}$$

and that we have an unphysical situation for y larger than this value.

Figure 4.2 shows $\Omega_{P/B}/\omega_s$ as a function of x and y in the legal region. We see that *over*-perfect dragging occurs. In figure 4.3 the perfect dragging line, where $\Omega_{P/B}/\omega_s = 1$, and the boundary of the region, $x = y^{-1/2}$ (representing the de Sitter horizon), are plotted in the (x, y)-plane. It is in the region between these two curves *over*-perfect dragging occurs. As $x \to \infty$ or $r_0 \gg 2M$ the two curves converge.

We must now ask whether it is possible to have situations where this unphysical phenomenon is present. We see that it occurs for $y \gtrsim 0.1$ (and for y very near the critical value). This means that

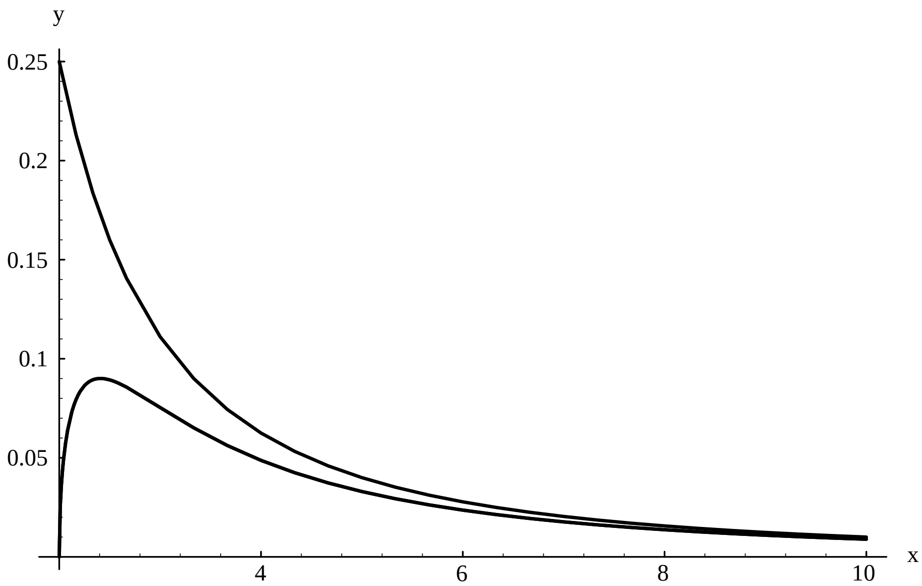
$$\frac{8\pi\rho M^2}{3} \gtrsim 0.1 \tag{4.227}$$

or

$$\rho \gtrsim \frac{3}{80\pi M^2} \gtrsim \frac{1}{100M^2}. \tag{4.228}$$

Restoring the numerical factors c and G, we obtain

$$\rho \gtrsim \frac{c^2}{100M^2G^3} \gtrsim 10^{80}\,\mathrm{kg/m^3}\left(\frac{1\,\mathrm{kg}}{M}\right)^2. \tag{4.229}$$



Figur 4.3: Perfect and over-perfect dragging

How large energy density is this? The so-called GUT-energy density, the one expected to be present at the beginning of the inflationary era, is approximately

$$\rho \sim 10^{79}\,\mathrm{kg/m^3}. \tag{4.230}$$

Hence, it is in theory possible to obtain energies large enough to achieve the unphysical over-perfect dragging effect. However, for small and thin shells (those considered here), the factor $1\,\mathrm{kg}/M$ becomes very large, and the situation will stay physical. Remember also that M is the *gravitational* mass of the spherical system, not the rest mass of the shell. In the cases considered here, the gravitational mass of the system will be smaller than the rest mass of the shell, because of the negative contribution of the vacuum energy inside the shell.

A spherical shell is often taken as a simplified cosmological model for our universe. At the present time we know that the universe is matter dominated, whence

$$\rho \ll \rho_{\mathrm{matter}} \sim 10^{-26}\,\mathrm{kg/m^3}. \tag{4.231}$$

According to (4.229) the spherical system must have an enormous gravitational mass in order to obtain over-perfect dragging:

$$\rho_{\mathrm{matter}} \gg \rho \sim 10^{80}\,\mathrm{kg/m^3}\left(\frac{1\,\mathrm{kg}}{M_{\mathrm{critical}}}\right)^2 \tag{4.232}$$

$$\Rightarrow \quad M_{\mathrm{critical}} \gg \rho_{\mathrm{matter}}^{-1/2}\cdot 10^{40}\,\mathrm{kg^{3/2}m^{-3/2}} \sim 10^{53}\,\mathrm{kg}. \tag{4.233}$$

As an estimate for the gravitating mass of the observable universe we may take

$$M_{\mathrm{universe}} \sim \frac{4}{3}\pi R_{\mathrm{universe}}^3\,\rho_{\mathrm{matter}} \tag{4.234}$$

where the radius is taken to be the age of the universe (10^{10} years) times the speed of light:

$$M_{\text{universe}} \sim 10^{52}\,\text{kg}. \tag{4.235}$$

Hence over-perfect dragging is impossible for this cosmological model.

It is therefore *reasonable* to argue that the above results are correct to a certain degree, and that the vacuum contributes to the dragging effects. The rotation of the vacuum can therefore be inferred by an observer because of its dragging properties. This is surprising since the motion of the vacuum cannot be detected directly because of its Lorenz-invariance.

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